The Emergence, Motion, and Disappearance of Magnetic Null Points

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SHINE Conference
Stowe, Vermont
July 6–10, 2015

Introduction

- Magnetic reconnection frequently occurs at and around null points: locations where the magnetic field strength equals zero
- Models of reconnection often assume symmetry such that the magnetic null point coincides with a flow stagnation point
- However, reconnection in nature and the laboratory is typically asymmetric (e.g., Cassak & Shay 2007; Murphy et al. 2010)
- Simulations of reconnection with asymmetry typically show a gap between the null and stagnation points
  - Consequently, there are often non-ideal flows across null points (e.g., Oka et al. 2008; Murphy 2010; Wyper & Jain 2013)
- In this poster, we:
  - Derive an exact expression for the 3D motion of null points
  - Discuss how non-ideal effects lead to flows across null points
  - Discuss the appearance and disappearance of null points
  - Show that an expression for the motion of a separator cannot be derived using solely local quantities

Definitions

- The time-dependent position of an isolated null point is

$$\mathbf{x}_n(t) \tag{1}$$

- The null point's velocity is:

$$\mathbf{U} \equiv \frac{\mathrm{d}\mathbf{x}_n}{\mathrm{d}t} \tag{2}$$

- The Jacobian matrix of the magnetic field at the null point is

$$\mathbf{M} \equiv \begin{pmatrix} \partial_x B_x & \partial_y B_x & \partial_z B_x \\ \partial_x B_y & \partial_y B_y & \partial_z B_y \\ \partial_x B_z & \partial_y B_z & \partial_z B_z \end{pmatrix}_{\mathbf{x}_n} \tag{3}$$

  The local linear magnetic field structure near the null is given by $\mathbf{B} = \mathbf{M} \cdot \delta\mathbf{x}$ where $\delta\mathbf{x} \equiv \mathbf{x} - \mathbf{x}_n(t)$

- We only consider null points because null lines and null planes are structurally unstable

We derive an expression for the motion of a null point in an arbitrary time-varying vector field with smooth derivatives

- First we take the derivative of the magnetic field following the motion of the magnetic field null,

$$\left.\frac{\partial \mathbf{B}}{\partial t}\right|_{\mathbf{x}_n} + \left.(\mathbf{U} \cdot \nabla)\,\mathbf{B}\right|_{\mathbf{x}_n} = 0 \tag{4}$$

  The RHS equals zero because the magnetic field will not change from zero as we follow the null point.
- Solving for **U** provides an expression for a null point's velocity

$$\mathbf{U} = -\mathbf{M}^{-1} \left.\frac{\partial \mathbf{B}}{\partial t}\right|_{\mathbf{x}_n} \tag{5}$$

  - Independent of Maxwell's equations
  - Assumes C^1 continuity of **B** about $\mathbf{x}_n$
  - Unique null point velocity when **M** is non-singular
- Geometric information about the magnetic field is contained within **M**. It is easier to change the position of a root of a function by a vertical shift if the slope is locally shallow.

We use Faraday's law to get an expression for the motion of a null point that remains independent of Ohm's law

- Faraday's law is given by

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{6}$$

- By applying Faraday's law to Eq. 5, we arrive at

$$\mathbf{U} = \mathbf{M}^{-1} \; \nabla \times \mathbf{E}\big|_{\mathbf{x}_n} \tag{7}$$

In resistive MHD, null point motion results from a combination of advection by the bulk plasma flow and resistive diffusion of the magnetic field

- Next, we apply the resistive MHD Ohm's law,

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} \tag{8}$$

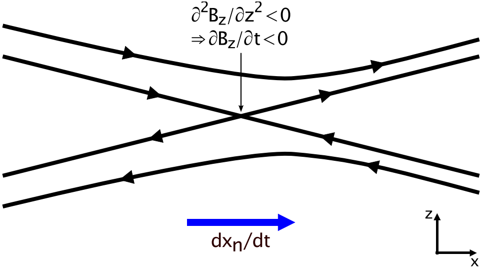
  where we assume the resistivity to be uniform.

- The expression for the rate of motion of a null point becomes

$$\mathbf{U} = \mathbf{V} - \eta\, \mathbf{M}^{-1}\, \nabla^2 \mathbf{B} \tag{9}$$

  where all quantities are evaluated at the magnetic null point. The terms on the RHS represent null point motion by
  - Bulk plasma flow
  - Resistive diffusion of the magnetic field

Murphy (2010): 1D X-line retreat via resistive diffusion



- B_z is negative above and below the X-line
- Diffusion of B_z leads to the current X-line position having negative B_z at a slightly later time
- The X-line moves to the right as a result of diffusion of the normal component of the magnetic field
- The 1D expression for X-line motion is:

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = V_x(x_n) - \eta \left[\frac{\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial z^2}}{\frac{\partial B_z}{\partial x}} \right]_{x_n} \tag{10}$$

Contribution from the generalized Ohm's law

- Additional terms in the generalized Ohm's law can be incorporated by re-evaluating Eq. 7.
- For example, if we choose our Ohm's law to be

$$\mathbf{E} + \mathbf{V}_i \times \mathbf{B} = \eta \mathbf{J} + \frac{\mathbf{J} \times \mathbf{B}}{n_e e} - \frac{\nabla P_e}{n_e e} \tag{11}$$

with $\mathbf{J} = n_e e \left(\mathbf{V}_i - \mathbf{V}_e\right)$, then Eq. 7 becomes

$$\mathbf{U} = \mathbf{V}_e - \eta \, \mathbf{M}^{-1} \, \nabla^2 \mathbf{B} + \mathbf{M}^{-1} \left(\frac{\nabla n_e \times \nabla P_e}{n_e^2 e} \right) \tag{12}$$

Again, all terms are evaluated at the null point.

  - The relevant plasma velocity becomes the electron velocity rather than the bulk plasma velocity.
  - The last term corresponds to the Biermann battery.

What does it mean for a magnetic null point to move?

- The velocity of a null point depends intrinsically on *local* plasma parameters evaluated at the null
- Global dynamics help set the local conditions
- A unique null point velocity exists if $\mathbf{M}$ is non-singular
- Nulls are not objects and cannot be pushed by, e.g., pressure gradient forces
    - Indirect coupling between the momentum equation and the combined Faraday/Ohm's law
    - Plasma not permanently affixed to nulls in non-ideal cases
- Our expression provides a further constraint on the structures of asymmetric diffusion regions (Cassak & Shay 2007)

Appearance and disappearance of null points

- As long as $\mathbf{M}$ is non-singular at the null, then there exists a unique null point velocity $\rightarrow$ linear null points are structurally stable (Greene 1988)
- In resistive MHD, nulls must diffuse in and out of existence
  - This provides geometric constraints that are not always accounted for during bifurcation theory and topological analysis
- Null points appear and disappear in pairs during the bifurcation of a degenerate null point with $\det \mathbf{M} = 0$
  - Typically, $\mathrm{rank}\, \mathbf{M} = 2$ and $\mathrm{nullity}\, \mathbf{M} = 1$ so that the null space of $\mathbf{M}$ is one-dimensional

Local magnetic field structure about a degenerate null

- During a bifurcation, it is necessary to include at least second order terms in the Taylor series expansion of $\mathbf{B}$:

$$\mathbf{B}(\delta\mathbf{x},\delta t) = \mathbf{M}\cdot\delta\mathbf{x} + \frac{\partial\mathbf{B}_n}{\partial t}\delta t + \frac{1}{2}\begin{bmatrix} \delta\mathbf{x}^T\,\mathbf{H}_x\,\delta\mathbf{x} \\ \delta\mathbf{x}^T\,\mathbf{H}_y\,\delta\mathbf{x} \\ \delta\mathbf{x}^T\,\mathbf{H}_z\,\delta\mathbf{x} \end{bmatrix} + \frac{\delta t}{2}\begin{bmatrix} (\delta\mathbf{x}\cdot\nabla)\partial_t B_x \\ (\delta\mathbf{x}\cdot\nabla)\partial_t B_y \\ (\delta\mathbf{x}\cdot\nabla)\partial_t B_z \end{bmatrix} + \frac{\delta t^2}{2}\frac{\partial^2\mathbf{B}_n}{\partial t^2} + \mathcal{O}(\|\delta\mathbf{x}\|^3,\delta t^3), \tag{13}$$

  where $\mathbf{H}_{k,ij} = \partial_i\partial_j B_k$ for $i,j,k\in\{x,y,z\}$ and all derivatives, Jacobians, and Hessians are evaluated at the null point
- While some of these terms are occasionally ignorable, it is vital to include second order terms along the null space of $\mathbf{M}$ and in the space between a bifurcating null-null pair

What is the instantaneous velocity of a bifurcating null-null pair?

- Consider a second order magnetic null point with $\mathrm{rank}\,\mathbf{M} = 2$ undergoing a bifurcation with $\dot{\mathbf{B}} \neq 0$
- Along the null space of $\mathbf{M}$, the magnetic field structure is second order. Each component of $\mathbf{B}$ has an extremum at the null.
  - The instantaneous component of velocity of separation or convergence along the null space of $\mathbf{M}$ will typically be infinite
- Next, consider the 2D plane that is orthogonal to the null space of $\mathbf{M}$. In this plane, the linear terms remain.
  - The instantaneous component of velocity of separation or convergence orthogonal to the null space of $\mathbf{M}$ will typically be finite

Can we perform a similar local analysis to describe the motion of separators?

- A separator is a magnetic field line connecting two null points
  - These are often important locations for reconnection.
- Suppose that there is non-ideal behavior only along one segment of a separator.
- At a slightly later time, the field line in the ideally evolving region will in general no longer be the separator, even though the evolution was locally ideal.
- Therefore, it is not possible to find an exact expression describing separator motion based solely on local parameters.
- However, a global approach could lead to an exact expression by taking into account connectivity changes along the separator as well as motion of its endpoints.

Separator motion in 2D

- Consider the motion of a magnetic field line connecting two null points in a 2D system
- The magnetic field is described using

$$\mathbf{B} = \nabla \times (A_z \hat{\mathbf{z}}) \tag{14}$$

  where A_z is the magnetic flux and field lines lie along contours of constant A_z
- Suppose two linear null points at $\mathbf{x}_a$ and $\mathbf{x}_b$ are connected by a field line at $t = 0$. Then $A_z(\mathbf{x}_a) = A_z(\mathbf{x}_b) \equiv A_s(t)$.
- Define $\mathbf{V}_\perp$ as the velocity of the separator orthogonal to the magnetic field
- The following method allows us to find $\mathbf{V}_\perp$ along the separator in 2D. We hope to extend this method to 3D to provide insight into the evolution and bifurcations of magnetic skeletons and non-ideal flows across separators.

Finding the velocity of a separator in 2D

- Find the null point velocities: $\mathbf{U}_a \equiv \dot{\mathbf{x}}_a$ and $\mathbf{U}_b \equiv \dot{\mathbf{x}}_b$
- Evaluate the LHS of the expression

$$\frac{\partial A_z}{\partial t} + \mathbf{U} \cdot \nabla A_z = \frac{\mathrm{d}A_n}{\mathrm{d}t} \tag{15}$$

  at $\mathbf{x}_a$ using $\mathbf{U}_a$ and at $\mathbf{x}_b$ using $\mathbf{U}_b$ to find the rate of change of magnetic flux at each null point: $\frac{\mathrm{d}A_a}{\mathrm{d}t}$ and $\frac{\mathrm{d}A_b}{\mathrm{d}t}$.
- If $\frac{\mathrm{d}A_a}{\mathrm{d}t} \neq \frac{\mathrm{d}A_b}{\mathrm{d}t}$, then the separator will not continue to exist
  - Separators in 2D are not structurally stable
- At each point along the separator, $\mathbf{V}_\perp$ can be calculated using

$$\frac{\partial A_z}{\partial t} + \mathbf{V}_\perp \cdot \nabla A_z = \frac{\mathrm{d}A_s}{\mathrm{d}t} \tag{16}$$

  and the requirement that $\mathbf{V}_\perp \cdot \mathbf{B} = 0$.
- This approach shows how local and global information are both necessary to calculate the velocity of a separator.

Conclusions

- We derive an exact expression for the motion of a magnetic null point that depends solely on parameters evaluated at the null. This expression can be applied for arbitrary Ohm's law.
- In resistive MHD, the position of a null point can change via bulk plasma flow or resistive diffusion of the magnetic field.
- Magnetic null points must diffuse in and out of existence through non-ideal effects. Upon formation, a new null is degenerate before bifurcating into a null-null pair. Second order terms are needed to describe these bifurcations.
- An expression for the motion of separators must include information on plasma motion and connectivity changes along its entire length.

Open Questions

- What is the interplay between small and large scales during null point motion?
- How do null points diffuse in and out of existence in realistic systems?
- How do we perform a general analysis of the bifurcation of second order null points?
- How do we find the initial orientation of separator(s) just after a null point bifurcation?
- How do we describe the motion of separators in 3D?
  - What is the nature of non-ideal flows across separators?
  - What is the nature of separator bifurcations?

- The inaugural Inclusive Astronomy was held last month. We ended up trending on Twitter! (#IA2015)
- While most equity and inclusion work has focused along a single dimension of identity (most often either gender, race, or LGBTQIA+ identity), the goal of this meeting was to take a multi-dimensional (*intersectional*) approach
- Some of the topics were: intersectionality, microaggressions, disability justice, accessibility, and racism in astronomy
- A key outcome of this meeting is the development of a set of inclusive astronomy recommendations
- The meeting website is at `http://vu.edu/ia2015`. Talks and presentation slides will be posted online soon!