Introduction to Magnetohydrodynamics

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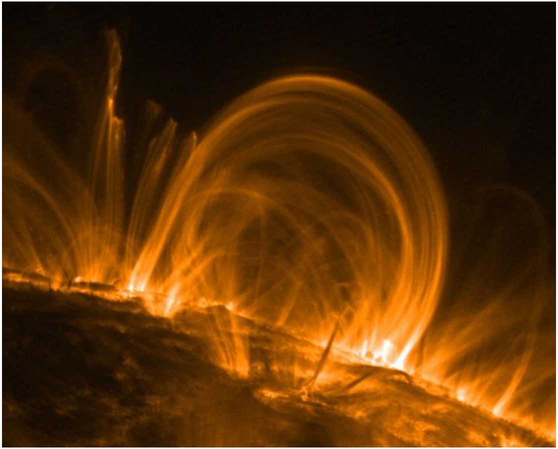
To MHD and beyond!

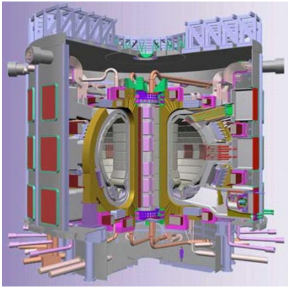
- What is MHD?
- The equations of MHD and their physical meaning
- Waves in MHD
  - Alfvén waves
  - Slow magnetosonic waves
  - Fast magnetosonic waves
- Beyond MHD
  - Extensions to MHD
  - Plasma kinetic theory
- Magnetic reconnection
- Final thoughts

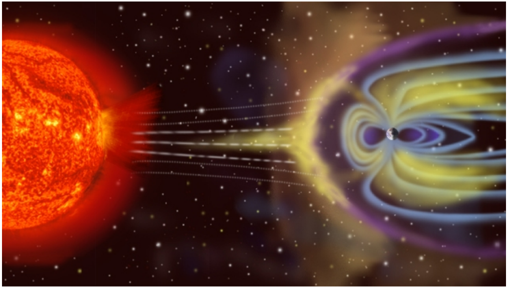
What is MHD?



- Magnetohydrodynamics (MHD) couples Maxwell's equations of electromagnetism with hydrodynamics to describe the macroscopic behavior of conducting fluids such as plasmas.

MHD is important in solar physics, astrophysics, space plasma physics, and in laboratory plasma experiments





Left: The International Thermonuclear Experimental Reactor (ITER; currently under construction)
Right: Interaction between the solar wind and the Earth's magnetosphere

MHD at a glance (SI units)

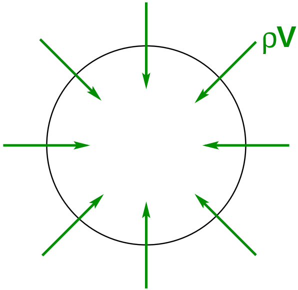
Continuity Equation	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Momentum Equation	$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla p$
Ampere's law	$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$
Faraday's law	$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$
Ideal Ohm's law	$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$
Divergence constraint	$\nabla \cdot \mathbf{B} = 0$
Adiabatic Energy Equation	$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p}{\rho^\gamma} \right) = 0$

Definitions: **B**, magnetic field; **V**, plasma velocity; **J**, current density; **E**, electric field; ρ, mass density; p, plasma pressure; γ, ratio of specific heats (usually 5/3); t, time.

The MHD approximation

- Assume the plasma behaves as a fluid
  - Macroscopic (low frequency, long wavelength) behavior
- Assume that the gyroradius is small
- Ignore the most significant physics advances since 1860:
  - Relativity ($v^2 \ll c^2$)
  - Quantum mechanics
  - Displacement current in Ampere's law
- Assume the plasma is fully ionized
  - Limited applicability to weakly ionized plasmas like the photosphere and chromosphere
- Assume collisions are frequent enough that the particle distribution function is Maxwellian
  - Not always true in the solar wind and laboratory plasmas
- Ideal MHD assumes no resistivity, viscosity, thermal conduction, or radiative cooling

The continuity equation describes conservation of mass



- The continuity equation written in conservative form is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

- The partial derivative $\partial \rho / \partial t$ refers to the change in density at a single point in space
- The divergence of the mass flux $\nabla \cdot (\rho \mathbf{V})$ says how much plasma goes in and out of the region
- Put sources and sinks of mass on RHS

The second golden rule of astrophysics

"The density of wombats

times the velocity of wombats

gives the flux of wombats."

The momentum equation is analogous to $m\mathbf{a} = \mathbf{F}$

- The momentum equation is

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla p$$

  Additional forces go on the right hand side (e.g., gravity).

- The total derivative is given by

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$$

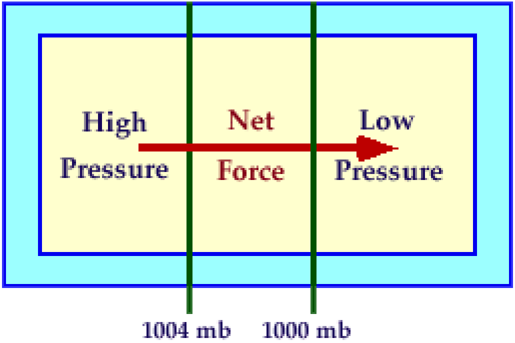
  and represents the derivative you take as you follow a parcel of plasma.

- In a static equilibrium:

$$\mathbf{J} \times \mathbf{B} = \nabla p$$

  When $\mathbf{J} \times \mathbf{B} = 0$, the plasma is 'force-free'

The pressure gradient force $-\nabla p$ pushes plasma from regions of high pressure to low plasma pressure



The Lorentz force term includes two components

- The current density is given by the relative drift between ions and electrons:

$$\mathbf{J} = ne\left(\mathbf{V}_i - \mathbf{V}_e\right)$$

$\rightarrow \mathbf{J} \times \mathbf{B}$ is analogous to $\mathbf{F} = q\mathbf{V} \times \mathbf{B}$.

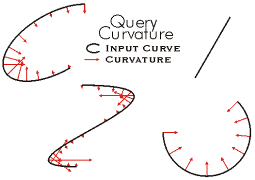
- Using vector identities and Ampere's law ($\mu_0\mathbf{J} = \nabla \times \mathbf{B}$), we rewrite the Lorentz force term $\mathbf{J} \times \mathbf{B}$ as:

$$\mathbf{J} \times \mathbf{B} \quad = \frac{\left(\mathbf{B} \cdot \nabla\right)\mathbf{B}}{\mu_0} - \nabla\left(\frac{B^2}{2\mu_0}\right)$$

However: the Lorentz force is orthogonal to $\mathbf{B}$, but these two terms are not.

The Lorentz force can be decomposed into two terms with forces othogonal to $\mathbf{B}$ using field line curvature

- The curvature vector $\boldsymbol{\kappa}$ points toward the center of curvature and gives the rate at which the tangent vector turns:

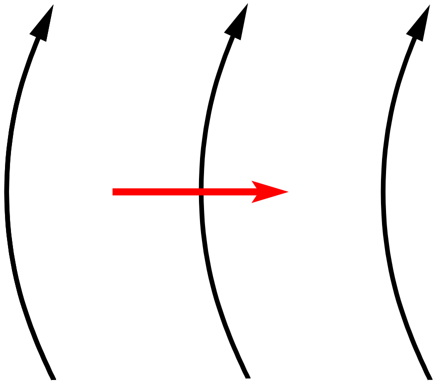


- We can then write the Lorentz force as

$$\underbrace{\mathbf{J} \times \mathbf{B}}_{\text{Lorentz force}} = \underbrace{\boldsymbol{\kappa} \frac{B^2}{\mu_0}}_{\text{magnetic tension}} - \underbrace{\nabla_\perp \left(\frac{B^2}{2\mu_0} \right)}_{\text{magnetic pressure}} \tag{1}$$

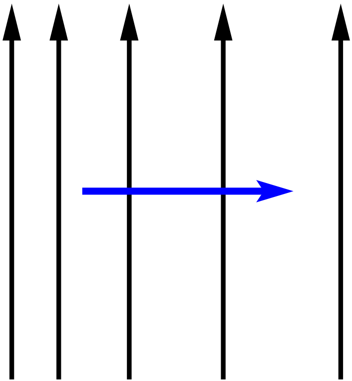
where all terms are orthogonal to $\mathbf{B}$. The operator $\nabla_\perp$ takes the gradient only in the direction orthogonal to $\mathbf{B}$.

The magnetic tension force wants to straighten magnetic field lines



- The tension force is directed radially inward with respect to magnetic field line curvature

Regions of high magnetic pressure exert a force towards regions of low magnetic pressure



- The magnetic pressure is given by $p_B \equiv \frac{B^2}{2\mu_0}$

The ratio of the plasma pressure to the magnetic pressure is an important dimensionless number

- Define plasma β as

$$\beta \equiv \frac{\text{plasma pressure}}{\text{magnetic pressure}} \equiv \frac{p}{B^2/2\mu_0}$$

- If $\beta \ll 1$ then the magnetic field dominates
  - Solar corona
- If $\beta \gg 1$ then plasma pressure forces dominate
  - Solar interior
- If $\beta \sim 1$ then pressure/magnetic forces are both important
  - Solar chromosphere
  - Parts of the solar wind and interstellar medium
  - Some laboratory plasma experiments

Faraday's law tells us how the magnetic field varies with time

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

But how do we get the electric field?

Ohm's law provides the electric field

- The ideal MHD Ohm's law is given by

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$$

- In ideal MHD, the magnetic field is *frozen-in* to the plasma. If two parcels of plasma are connected by a magnetic field line at one time, then they will be connected by a magnetic field line at all other times.
- For resistive MHD, Ohm's law becomes

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

  where η is the resistivity. Resistivity allows the frozen-in condition to be broken.
- Can also include the Hall effect which is important on short length scales

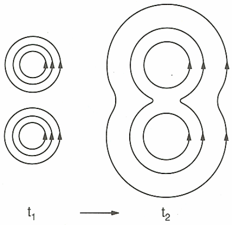
With Ohm's law we can rewrite Faraday's law as the induction equation

- Using the resistive Ohm's law:

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{V} \times \mathbf{B})}_{\text{convection}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}}_{\text{diffusion}}$$

  Diffusion is usually represented by a second order spatial derivative.

- An example of resistive diffusion:



Thermal conduction

- Thermal conduction is a common extension to MHD
- Heat diffuses much more quickly along magnetic field lines than orthogonal to them
- Anisotropic thermal conduction is a challenge in numerical simulations
- The temperature along magnetic field lines is usually approximately constant
    - Exceptions: when localized heating occurs on short timescales, or there are rapid connectivity changes

Waves

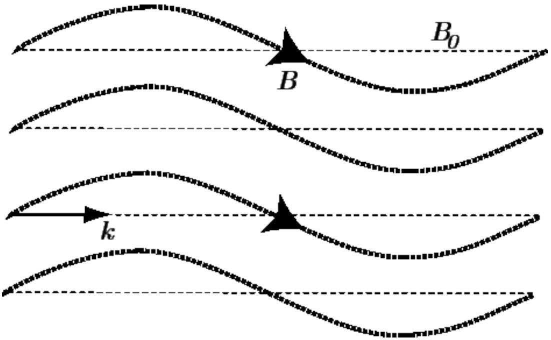
- There are three primary waves that arise from MHD:
  - Alfvén wave
  - Slow magnetosonic wave
  - Fast magnetosonic wave
- There are two important speeds
  - The sound speed is given by

$$C_s \equiv \sqrt{\frac{\gamma p}{\rho}}$$

  - The Alfvén speed is given by

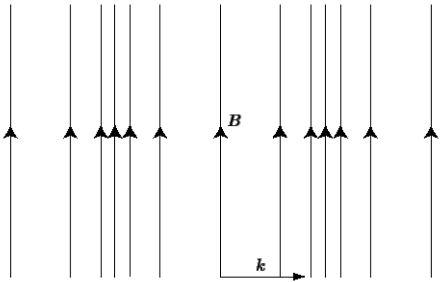
$$V_A \equiv \frac{B}{\sqrt{\mu_0 \rho}}$$

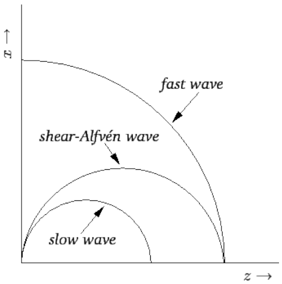
Alfvén Waves



- Alfvén waves propagate at the Alfvén speed, $V_A \equiv B/\sqrt{\mu_0 \rho}$
- The restoring force is magnetic tension
- This is a shear wave with no compression involved
- Disturbances propagate parallel to **B**

Slow and Fast Magnetosonic Waves





- *Left:* The restoring forces for magnetosonic waves propagating perpendicular to **B** are given by gas and magnetic pressure gradients. This shows a compressional wave.
- *Right:* The phase velocity of MHD waves as a function of angle when **B** is in the *z* direction and β is small.
- Sound waves are magnetosonic waves propagating along **B**

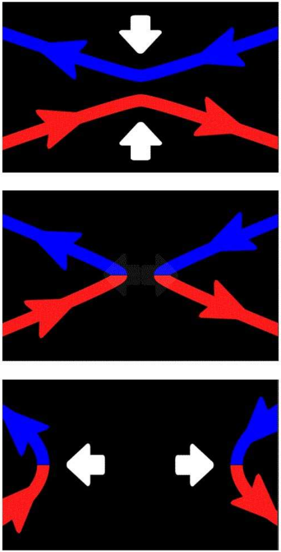
How useful is MHD?

- MHD is appropriate for large-scale, low-frequency behavior
- MHD is a good predictor of stability
  - Non-MHD effects sometimes stabilize or destabilize...
- MHD is often inappropriate when there are non-Maxwellian distribution functions
  - Including in collisionless plasmas or when there are energetic, non-thermal particles
- MHD is a reasonable approximation for most solar physics applications, but there are many effects beyond MHD that will often be important
- MHD usually does not usually work well for laboratory plasmas

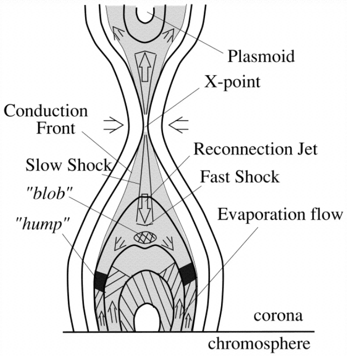
There are two general approaches to going beyond MHD

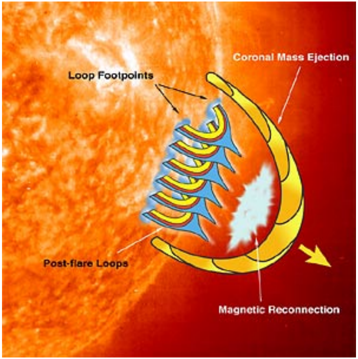
- *Extended MHD.* Keep the fluid approximation but include effects such as
  - Anisotropic thermal conduction
  - Different ion and electron temperatures: $T_{ion} \neq T_{electron}$
  - Different pressures parallel and perpendicular to $\mathbf{B}$: $p_\perp \neq p_\parallel$
  - Finite Larmor radius effects
  - Hall effect
- *Kinetic theory.* Abandon the fluid approximation and keep track of particle distribution function.
  - Computationally intensive but often necessary to include important physical processes
- Sometimes we can take a hybrid between these two approaches

Magnetic Reconnection is the breaking and rejoining of magnetic field lines in a highly conducting plasma



Solar flares and CMEs are powered by magnetic reconnection





- Explosive release of magnetic energy
- Bidirectional Alfvénic jets
- Very efficient particle acceleration
- Flux ropes escape as coronal mass ejections (CMEs)

Magnetic reconnection is a fundamental process in laboratory and astrophysical plasmas

- Classical theories based on resistive diffusion predict slow reconnection (weeks to months...)
- *Fast* reconnection allows magnetic energy to be explosively converted into kinetic and thermal energy
- Collisionless or non-fluid effects are (probably) needed to explain why fast reconnection occurs in flares (tens of seconds to minutes!)

Summary

- MHD describes the macroscopic behavior of plasmas
- Each term in the MHD equations represents a different physical effect
- There are three types of MHD waves: Alfvén waves, fast magnetosonic waves, and slow magnetosonic waves
- Often, physics beyond MHD is needed to describe plasma physics phenomena
- Magnetic reconnection is the breaking and rejoining of magnetic field lines in a highly conducting plasma
  - Releases magnetic energy during solar flares and CMEs
  - Degrades confinement in laboratory plasmas

Useful references

- *The Physics of Plasmas* by T.J.M. Boyd and J.J. Sanderson. One of the most understandable introductions to plasma physics that I've found.
- *Magnetohydrodynamics of the Sun* by Eric Priest. Very useful resource for the mathematical properties of MHD as applied to the Sun.
- *Principles of Magnetohydrodynamics* by Hans Goedbloed and Stefaan Poedts. Good introduction to MHD with a broad focus on applications.
- *Ideal Magnetohydrodynamics* by Jeffrey Freidberg. Very good out-of-print introduction to MHD in particular. Later chapters focus more on laboratory plasmas.
- *Introduction to Plasma Physics and Controlled Fusion* by Francis Chen. A beginning graduate level introduction to plasma physics. Less emphasis on MHD.