MHD Instabilities

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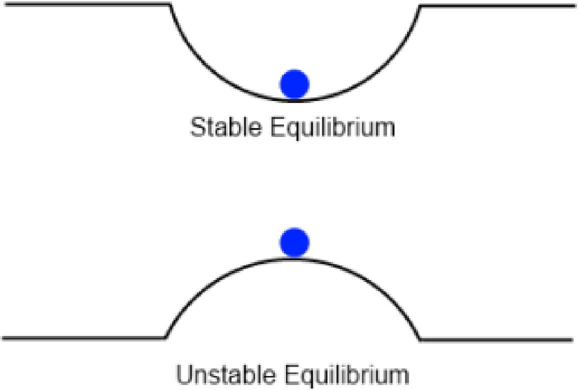
These slides are based off of Boyd & Sanderson's *The Physics of Plasmas*, Schnack's *Lectures in Magnetohydrodynamics*, Kulsrud's *Plasma Physics for Astrophysics*, Freidberg's *Ideal Magnetohydrodynamics*, and slides from Andrei Simakov. The discussion on why $\mathbf{k} \cdot \mathbf{B}_0 = 0$ surfaces are important can largely be attributed to Tom Bird.

Outline

- Overview of instabilities
- Initial value formulation
- Normal mode formulation
- Variational formulation
- Energy principle
- The magnetic Rayleigh-Taylor instability

Basic idea behind physical instabilities



- If you perturb the stable equilibrium, the ball will oscillate around the equilibrium position
- If you perturb the unstable equilibrium, the ball will roll away

Let's take this analogy further

- If $W(x)$ is the potential energy, then the particle feels a force:

$$F_x = -\frac{\partial W}{\partial x} \tag{1}$$

- Equilibria occur when

$$\frac{\partial W}{\partial x} = 0 \tag{2}$$

- To lowest order in the Taylor expansion about an equilibrium point, the change in potential energy is

$$\delta W = \frac{1}{2}\left(\frac{\partial^2 W}{\partial x^2}\right)(\Delta x)^2 \tag{3}$$

- The equilibria are stable when $\delta W > 0$
- The equilibria are unstable when $\delta W < 0$

Why study plasma instabilities?

- Plasmas are host to numerous instabilities
- An equilibrium will not exist for long if it is unstable
- Plasma instabilities play important roles in a variety of astrophysical phenomena
  - Star formation/accretion disks
  - Particle acceleration
  - Interstellar and intracluster mediums
  - Solar/stellar eruptions
- Plasma instabilities are a source of turbulence which in turn can drive transport
  - Magnetorotational instability
  - Various short wavelength instabilities in fusion devices
- In magnetically confined fusion plasmas, instabilities are a key barrier to good confinement
- The universe would be a very different place if not for plasma instabilities

One could easily fill a full course on plasma instabilities

List of plasma instabilities [edit]

- Bennett pinch instability (also called the z-pinch instability)
- Beam acoustic instability
- Bump-in-tail instability
- Buneman instability,[2]
- Cherenkov instability,[3]
- Chute instability
- Coalescence instability,[4]
- Collapse instability
- Counter-streaming instability
- Cyclotron instabilities, including:
  - Alfven cyclotron instability
  - Electron cyclotron instability
  - Electrostatic ion cyclotron Instability
  - Ion cyclotron instability
  - Magnetoacoustic cyclotron instability
  - Proton cyclotron instability
  - Nonresonant Beam-Type cyclotron instability
  - Relativistic ion cyclotron instability
  - Whistler cyclotron instability
- Diocotron instability,[5] (similar to the Kelvin-Helmholtz fluid instability).
- Disruptive instability (in tokamaks)
- Double emission instability
- Drift wave instability
- Edge-localized modes [6]
- Electrothermal instability
- Farley-Buneman instability,[7]
- Fan instability
- Filamentation instability
- Firehose instability (also called Hose instability)
- Flute instability
- Free electron maser instability
- Gyrotron instability
- Helical instability (helix instability)
- Helical kink instability
- Hose instability (also called Firehose instability)
- Interchange instability
- Ion beam instability
- Kink instability
- Lower hybrid (drift) instability (in the Critical ionization velocity mechanis
- Magnetic drift instability
- Magnetorotational instability (in accretion disks)
- Magnetothermal instability (Laser-plasmas) [8]
- Modulation instability
- Non-Abelian instability (see also Chromo-Weibel instability)
- Chromo–Weibel instability
- Non-linear coalescence instability
- Oscillating two stream instability, see two stream instability
- Pair instability
- Parker instability (magnetic buoyancy instability)
- Peratt instability (stacked toroids)
- Pinch instability
- Sausage instability
- Slow Drift Instability
- Tearing mode instability
- Two-stream instability
- Weak beam instability
- Weibel instability
- z-pinch instability, also called Bennett pinch instability

Why MHD?

- Ideal MHD is frequently used to describe macroscopic instabilities
- MHD is often a mediocre approximation, but it does a surprisingly good job at describing instabilities
- MHD captures most of the essential physics of force balance
  - Magnetic tension, magnetic pressure, plasma pressure
- MHD instabilities capture the most explosive behavior
- Instability theory is closely related to wave theory
- However, ideal MHD does not always give the right picture
  - There are some laboratory configurations that are stable despite being unstable in ideal MHD
- In astrophysics, heat conduction and radiative cooling can also be destabilizing

Ideal, resistive, and kinetic instabilities

- Ideal instabilities are usually the strongest and most unstable
  - Current-driven vs. pressure-driven
- Resistive instabilities are stable unless $\eta \neq 0$
  - Growth rate is usually slower than ideal instabilities
  - Often associated with magnetic reconnection
- Kinetic instabilities are often microinstabilities that occur when the distribution functions are far from Maxwellian
  - Two-stream, Weibel, Buneman, firehose, etc.
  - Important for near-Earth space plasmas, laboratory plasmas, cosmic ray interactions with ambient plasma, and dissipation of turbulence

General strategy for studying plasma stability

- Start from an initial equilibrium:

$$\frac{\mathbf{J}_0 \times \mathbf{B}_0}{c} = \nabla p_0 \tag{4}$$

- Linearize the equations of MHD
- Slightly perturb that equilibrium
  - Analytically or numerically
- If there is a growing perturbation, the system is unstable
- If no growing perturbation exists, the system is stable
- Use a combination of numerical simulations, experiments, and observations to study nonlinear dynamics

Linearizing the equations of ideal MHD

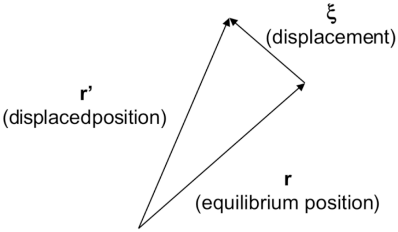
- Follow the procedure for waves and represent fields as the sum of equilibrium ('0') and perturbed ('1') components

$$
\begin{aligned}
\rho(\mathbf{r}, t) &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t) && (5)\\
\mathbf{V}(\mathbf{r}, t) &= \mathbf{V}_1(\mathbf{r}) && (6)\\
p(\mathbf{r}, t) &= p_0(\mathbf{r}) + p_1(\mathbf{r}, t) && (7)\\
\mathbf{B}(\mathbf{r}, t) &= \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t) && (8)
\end{aligned}
$$

  The perturbed fields are much smaller than the equilibrium fields, and $\mathbf{V}_0 = 0$ in the absence of background flow
- To zeroeth order, a static equilibrium is given by

$$\nabla p_0 = \frac{(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0}{4\pi} \qquad (9)$$

Recall that the displacement vector $\boldsymbol{\xi}$ describes how much the plasma is displaced from an equilibrium



- If $\boldsymbol{\xi}(\mathbf{r}, t = 0) = 0$, then the displacement vector is

$$\boldsymbol{\xi}(\mathbf{r}, t) \equiv \int_0^t \mathbf{V}_1(\mathbf{r}, t')\mathrm{d}t' \tag{10}$$

- Its time derivative is just the perturbed velocity,

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{V}_1(\mathbf{r}, t) \tag{11}$$

The linearized equations in terms of the displacement vector, $\boldsymbol{\xi}$

- The linearized continuity, induction, energy, and momentum equations are

$$\begin{aligned}
\rho_1(\mathbf{r},t) &= -\boldsymbol{\xi}(\mathbf{r},t)\cdot\nabla\rho_0 - \rho_0\nabla\cdot\boldsymbol{\xi}(\mathbf{r},t) && (12)\\
\mathbf{B}_1(\mathbf{r},t) &= \nabla\times\left[\frac{\boldsymbol{\xi}(\mathbf{r},t)\times\mathbf{B}_0(\mathbf{r})}{c}\right] && (13)\\
p_1(\mathbf{r},t) &= -\boldsymbol{\xi}(\mathbf{r},t)\cdot\nabla p_0(\mathbf{r}) - \gamma p_0(\mathbf{r})\nabla\cdot\boldsymbol{\xi}(\mathbf{r},t) && (14)\\
\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} &= \mathbf{F}[\boldsymbol{\xi}(\mathbf{r},t)] && (15)
\end{aligned}$$

where the MHD force operator is

$$\begin{aligned}
\mathbf{F}(\boldsymbol{\xi}) &= \nabla\left(\boldsymbol{\xi}\cdot\nabla p_0 + \gamma p_0\nabla\cdot\boldsymbol{\xi}\right)\\
&\quad+\frac{1}{4\pi}\left(\nabla\times\mathbf{B}_0\right)\times\left[\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\right]\\
&\quad+\frac{1}{4\pi}\left\{\left[\nabla\times\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\right]\times\mathbf{B}_0\right\} && (16)
\end{aligned}$$

Building up intuition for the force operator

- If $\boldsymbol{\xi} \cdot \mathbf{F} < 0$:
  - The displacement and force are in opposite directions
  - The force opposes displacements
  - The system will typically oscillate around the equilibrium
  - This corresponds to a stable perturbation
- If $\boldsymbol{\xi} \cdot \mathbf{F} > 0$:
  - The displacement and force are in the same direction
  - The force encourages displacements
  - The perturbation will grow
  - This corresponds to an unstable perturbation
- If $\boldsymbol{\xi} \cdot \mathbf{F} = 0$, then this perturbation is neutrally stable
- MHD does not allow overstable solutions where the restoring force would be strong enough to overcorrect for and amplify the oscillations
  - These would require an energy sink or source

Initial value formulation

- MHD stability can be investigated as an initial value problem by finding numerical or analytical solutions to

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}[\boldsymbol{\xi}(\mathbf{r}, t)] \tag{17}$$

  with appropriate initial and boundary conditions.
- Numerical simulations are particularly useful for
  - Complicated initial conditions
  - Understanding the linear and nonlinear evolution of a system
  - Finding the structure of the fastest growing mode
- While numerical simulations are very useful, analytical theory offers complementary (and often easier!) ways to determine whether or not a system is stable

Finding the normal mode solution

- Separate the space and time dependences:

$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(\mathbf{r}) T(t) \tag{18}$$

- The momentum equation becomes two decoupled equations:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -\omega^2 T \tag{19}$$

$$-\omega^2 \rho_0 \boldsymbol{\xi}(\mathbf{r}) = \mathbf{F}[\boldsymbol{\xi}(\mathbf{r})] \tag{20}$$

so that $T(t) = e^{i\omega t}$ and the solution is of the form

$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(\mathbf{r}) e^{i\omega t} \tag{21}$$

- $\boldsymbol{\xi}(\mathbf{r})$ is an eigenfunction of $\mathbf{F}$ with eigenvalue $-\omega^2 \rho_0$
- The BCs determine the permitted values of ω^2
  - These can be a discrete or continuous set

Finding the normal mode solution

- For a discrete set of ω^2, the solution is

$$\boldsymbol{\xi}(\mathbf{r}, t) = \sum_n \boldsymbol{\xi}_n(\mathbf{r}) e^{i\omega t} \tag{22}$$

  where $\boldsymbol{\xi}_n$ is the normal mode corresponding to its normal frequency ω_n
- Because $\mathbf{F}$ is self-adjoint, ω_n^2 must be real
- If $\omega_n^2 > 0$ for all n, then the equilibrium is stable
- If $\omega_n^2 < 0$ for any n, then the equilibrium is unstable
- Stability boundaries occur when $\omega = 0$
- Now all we have to do is solve for a possibly infinite number of solutions!

The MHD force operator is **self-adjoint**[1]

- An adjoint is a generalized complex conjugate for a functional
- The operator $\mathbf{F}(\boldsymbol{\xi})$ is **self-adjoint**. For any allowable displacement vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$

$$\int \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi})\, \mathrm{d}\mathbf{r} = \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta})\, \mathrm{d}\mathbf{r} \tag{23}$$

- Self-adjointness is related to conservation of energy
  - If there is dissipation, $\mathbf{F}$ will not be self-adjoint
- We need this property to prove that ω^2 is real.

[1] A proof is given by Freidberg (1987)

Only purely exponential or oscillatory solutions are allowed

- Dot $-\omega^2\rho\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$ with $\boldsymbol{\xi}^*$, and integrate over volume:

$$\omega^2 \int \rho_0 |\boldsymbol{\xi}|^2 \mathrm{d}\mathbf{r} = -\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) \mathrm{d}\mathbf{r} \tag{24}$$

- Dot the complex conjugate equation $-(\omega^*)^2\rho_0\boldsymbol{\xi}^* = \mathbf{F}(\boldsymbol{\xi}^*)$ with $\boldsymbol{\xi}$, and integrate over volume:

$$(\omega^*)^2 \int \rho_0 |\boldsymbol{\xi}|^2 \mathrm{d}\mathbf{r} = -\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}^*) \tag{25}$$

- Subtract the two equations and use that $\mathbf{F}$ is self-adjoint

$$\left[\omega^2 - (\omega^*)^2\right] \int \rho_0 |\boldsymbol{\xi}|^2 = 0 \tag{26}$$

  Only satisfied if ω^2 and $\boldsymbol{\xi}$ are real. [Need $\omega^2 = (\omega^*)^2$]

- Solutions are either **exponential** or **oscillatory**!

Showing that the normal modes are orthogonal

- Consider two discrete modes $(\boldsymbol{\xi}_m, \omega_m^2)$ and $(\boldsymbol{\xi}_n, \omega_n^2)$

$$-\omega_m^2 \rho_0 \boldsymbol{\xi}_m = \mathbf{F}(\boldsymbol{\xi}_m) \tag{27}$$

$$-\omega_n^2 \rho_0 \boldsymbol{\xi}_n = \mathbf{F}(\boldsymbol{\xi}_n) \tag{28}$$

- Dot the $\boldsymbol{\xi}_m$ equation with $\boldsymbol{\xi}_n$ and vice versa, integrate over volume, subtract, and use self-adjointedness of $\mathbf{F}$ to get

$$\left(\omega_m^2 - \omega_n^2\right) \int \rho_0 \, \boldsymbol{\xi}_m \cdot \boldsymbol{\xi}_n \, \mathrm{d}\mathbf{r} = 0 \tag{29}$$

If $\omega_m^2 \neq \omega_n^2$ (e.g., the modes are discrete) then

$$\int \rho_0 \, \boldsymbol{\xi}_m \cdot \boldsymbol{\xi}_n \, \mathrm{d}\mathbf{r} = 0 \tag{30}$$

The modes are **orthogonal** with weight function ρ_0!

Normal mode formulation

- This method requires less effort than the initial value formulation
- This method is more amenable to analysis
- This method cannot be used to describe nonlinear evolution (after the linearization approximation breaks down)
- Normal mode analysis requires that the eigenvalues are discrete and distinguishable
- However, there is a more elegant way to determine whether or not a system is stable
  - The energy principle!

The variational principle is the basis for the energy principle

- The kinetic energy is $\frac{1}{2}\rho_0\dot{\xi}^2$ integrated over the volume:

$$\begin{aligned} K &= \frac{1}{2}\int \rho_0 \dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\,\mathrm{d}\mathbf{r} \\ &= \frac{-\omega^2}{2}\int \rho_0 \boldsymbol{\xi}\cdot\boldsymbol{\xi}\,\mathrm{d}\mathbf{r} \\ &= \frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\,\mathrm{d}\mathbf{r} \end{aligned} \tag{31}$$

- The change in potential energy must then be $-K$:

$$\delta W = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\,\mathrm{d}\mathbf{r} \tag{32}$$

$\boxed{\text{If } \delta W < 0 \text{ for any perturbation, then the system is unstable.}}$

Strategy for the variational principle

- Choose a trial function

$$\boldsymbol{\xi} = \sum_n a_n \phi_n \tag{33}$$

where ϕ_n are a suitable choice of basis functions subject to the normalization condition

$$K = \text{const.} \tag{34}$$

- Minimize δW with respect to the coefficients a_n
- A lower bound for the growth rate γ is

$$\gamma \geq \sqrt{-\frac{\delta W}{K}} \tag{35}$$

The energy principle

- Generally we care more about whether or not a configuration is stable than finding the linear growth rate
- The linear growth stage quickly becomes overwhelmed by nonlinear effects
- The energy principle allows us to determine stability but at the cost of losing information about the growth rate
    - We lose the restriction regarding the normalization condition

Deriving the energy principle[2]

- Start from

$$\delta W = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\mathrm{d}\mathbf{r} \tag{36}$$

  (Think: work equals force times distance)

- Express δW as the sum of the changes in the potential energy of the plasma (δW_P), the surface (δW_S), and the associated vacuum solution (δW_V):

$$\delta W = \delta W_P + \delta W_S + \delta W_V \tag{37}$$

- The vacuum magnetic energy corresponds to the potential field solution for a given set of boundary conditions

[2]For a full derivation, refer to Freidberg (1987).

Deriving the energy principle

- The plasma, surface, and vacuum contributions are

$$
\begin{aligned}
\delta W_P &= \frac{1}{2}\int\left[\frac{B_1^2}{4\pi} - \boldsymbol{\xi}\cdot\left(\frac{\mathbf{J}_0\times\mathbf{B}_1}{c}\right) - p_1\left(\nabla\cdot\boldsymbol{\xi}\right)\right]\mathrm{d}\mathbf{r} \qquad (38)\\
\delta W_S &= \oint_{\mathcal{S}}\left(\boldsymbol{\xi}\cdot\hat{\mathbf{n}}\right)^2\left[\nabla\left(p_0+\frac{B_0^2}{8\pi}\right)\right]_1^2\cdot\mathrm{d}\mathbf{S} \qquad (39)\\
\delta W_V &= \int\frac{B_{1,vac}^2}{8\pi}\mathrm{d}\mathbf{r} \qquad (40)
\end{aligned}
$$

- In the case of a deformed boundary, all terms may exist. This corresponds to an **external** (free-boundary) mode.
- In the case of a fixed boundary, $\delta W_S = \delta W_V = 0$. This corresponds to an **internal** (fixed-boundary) mode.

The intuitive form of energy principle

- The energy principle can be written as

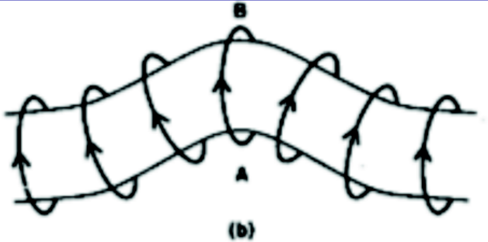
$$
\begin{aligned}
\delta W_P \;&=\; \frac{1}{2}\int \mathrm{d}\mathbf{r}\,\Bigg[\frac{|\mathbf{B}_{1\perp}|^2}{4\pi} + \frac{B^2}{4\pi}\left|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 \\
&\qquad + \gamma p\,|\nabla\cdot\boldsymbol{\xi}|^2 - 2\left(\boldsymbol{\xi}_\perp\cdot\nabla p\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right) \\
&\qquad - J_\parallel\left(\boldsymbol{\xi}_\perp^*\times\mathbf{b}\right)\cdot\mathbf{B}_{1\perp}\Bigg] \qquad (41)
\end{aligned}
$$

- The first three terms are always stabilizing (in order):
  - Energy required to bend field lines (shear Alfvén wave)
  - Energy necessary to compress **B** (compr. Alfvén wave)
  - The energy required to compress the plasma (sound wave)
- The remaining two terms can be stabilizing or destabilizing:
  - Pressure-driven (interchange) instabilities (related to $\mathbf{J}_\perp$, ∇p)
  - Current-driven (kink) instabilities (related to $\mathbf{J}_\parallel$)

Strategy for using the energy principle

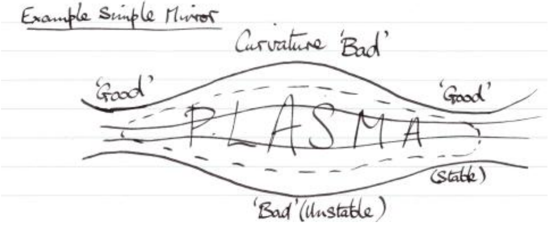
- The energy principle can be used with the same strategy as the variational principle
- If there exists a trial solution $\boldsymbol{\xi}$ for which $\delta W < 0$, then the configuration is unstable
- Information on the growth rate and linear structure of the instability is lost
- However, the energy principle helps identify the drivers of a particular instability

The kink instability



- Magnetic pressure is increased at point A where the perturbed field lines are closer together
- Magnetic pressure is decreased at point B where the perturbed field lines are separated
- A magnetic pressure differential causes the perturbation to grow
- The kink instability could be stabilized by magnetic field along the axis of the flux rope
  - Tension becomes a restoring force
- Usually long wavelengths are more unstable

Good curvature vs. bad curvature

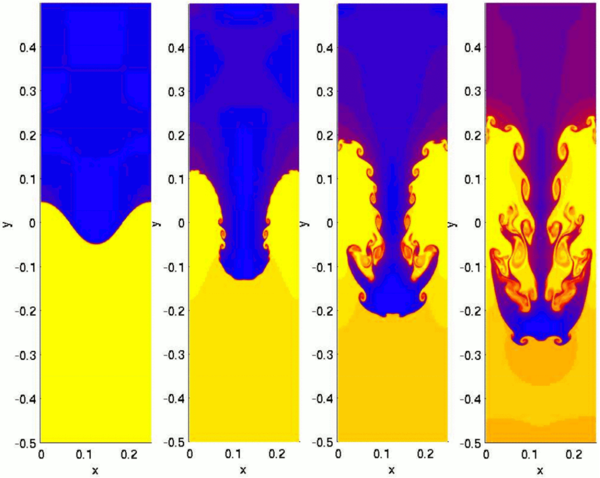


- Pressure-driven or interchange instabilities occur when

$$\boldsymbol{\kappa} \cdot \nabla p > 0 \tag{42}$$

  where $\boldsymbol{\kappa} \equiv \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$ is the curvature vector. This is a necessary but not sufficient criterion for pressure-driven instabilities.
- Instability occurs when it is energetically favorable for the magnetic field and plasma to switch places
- Usually short wavelengths are most unstable

The Rayleigh-Taylor instability occurs when dense fluid sits on top of sparse fluid



- An example of interchange behavior with nonlinear dynamics and secondary instabilities
- Magnetized equivalent: light fluid $\rightarrow$ magnetic field

Performing a normal mode analysis for the Rayleigh-Taylor instability[3]

- Choose an equilibrium magnetic field in the x-z plane that varies in the y direction

$$\mathbf{B}_0(y) = B_x(y)\,\hat{\mathbf{x}} + B_z(y)\,\hat{\mathbf{z}} \tag{43}$$

- Let the gravitational acceleration be

$$\mathbf{g} = -g\,\hat{\mathbf{y}} \tag{44}$$

- Choose a perturbation of the form

$$\boldsymbol{\xi}(\mathbf{r}, t) = [\xi_y(y)\,\hat{\mathbf{y}} + \xi_z(y)\,\hat{\mathbf{z}}]\, e^{i(kz-\omega t)} \tag{45}$$

[3]Following Boyd & Sanderson §4.7.1

Start from the momentum equation in terms of $\boldsymbol{\xi}$ and $\mathbf{F}$

- The momentum equation is

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi}(\mathbf{r}, t)) \tag{46}$$

- The force operator with gravity becomes

$$\begin{aligned}
\mathbf{F}(\boldsymbol{\xi}(\mathbf{r}, t)) &= \nabla\left(\boldsymbol{\xi} \cdot \nabla p_0\right) - \boldsymbol{\xi} \cdot \nabla \rho_0 \, \mathbf{g} \\
&\quad + \frac{1}{4\pi}\left(\nabla \times \mathbf{B}\right) \times \left[\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)\right] \\
&\quad + \frac{1}{4\pi}\left\{\left[\nabla \times \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)\right] \times \mathbf{B}_0\right\}
\end{aligned} \tag{47}$$

where we assume incompressibility:

$$\nabla \cdot \boldsymbol{\xi} = \frac{\mathrm{d}\xi_y}{\mathrm{d}y} + ik\xi_z = 0 \tag{48}$$

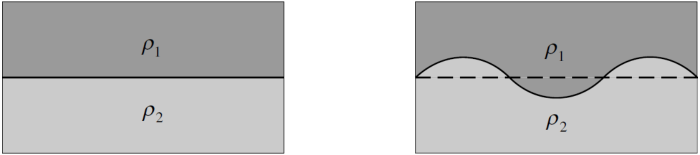
Finding a relation describing the Rayleigh-Taylor instability

- The momentum equation becomes the differential equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}y}\left[\left(\rho_0\omega^2 - \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{4\pi}\right)\frac{\mathrm{d}\xi_y}{\mathrm{d}y}\right] - k^2\left(\rho_0\omega^2 - \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{4\pi}\right)\xi_y - k^2 g\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\xi_y \quad (49)$$

  where we use gravity and incompressibility in the force operator
- This equation contains all the information we need to describe the Rayleigh-Taylor instability
- Next we consider different limits and configurations

The hydrodynamic limit of the Rayleigh-Taylor instability



- Set $\mathbf{B}_0 = 0$
- Assume that $\rho(y > 0) = \rho_1$ and $\rho(y < 0) = \rho_2$
- In each fluid, the differential equation reduces to

$$0 = \frac{\mathrm{d}^2\xi_y}{\mathrm{d}y} - k^2 \frac{\mathrm{d}\xi_y}{\mathrm{d}y} \tag{50}$$

with solutions

$$\xi_y(y > 0) = \xi_y(0)e^{-ky} \tag{51}$$

$$\xi_y(y < 0) = \xi_y(0)e^{ky} \tag{52}$$

The dispersion relationship for the hydrodynamic limit

- Integrating over the interval $-\epsilon < y < \epsilon$, letting $\epsilon \to 0$, and rearranging yields the dispersion relationship

$$\omega^2 = -\frac{kg\left(\rho_1 - \rho_2\right)}{\rho_1 + \rho_2} \tag{53}$$

- Instability happens for $\omega^2 < 0$, which is when the denser fluid is on top of the less dense fluid
- The growth rate is fastest for $\rho_1 \gg \rho_2$, and for short wavelengths
- In real systems, surface tension and viscosity will stabilize very short wavelengths modes

A uniform magnetic field stabilizes short wavelengths

- If we add in a uniform magnetic field parallel to the interface, then the dispersion relationship becomes

$$\omega^2 = -\frac{kg\,(\rho_1 - \rho_2)}{\rho_1 + \rho_2} + \frac{2\,(\mathbf{k} \cdot \mathbf{B}_0)^2}{4\pi\,(\rho_1 + \rho_2)} \tag{54}$$

- The term proportional to $(\mathbf{k} \cdot \mathbf{B})^2$ is stabilizing except when $\mathbf{k} \cdot \mathbf{B} \neq 0$
- If θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$, then instability only occurs when

$$k \geq \frac{2\pi g\,(\rho_1 - \rho_2)}{B_0^2 \cos^2\theta} \tag{55}$$

Short wavelength perturbations are stabilized by the magnetic field.

Unmagnetized plasma supported by a magnetic field

- Consider a configuration where
  - Top: density is ρ_1 but $\mathbf{B}_0 = 0$
  - Bottom: density is $\rho_2 \to 0$ but $\mathbf{B}_0$ provides support
- The dispersion relationship becomes

$$\omega^2 = -kg + \frac{(\mathbf{k} \cdot \mathbf{B})^2}{4\pi\rho_1} \tag{56}$$

- If instead there is a rotating magnetic field parallel to the interface, then instability will only occur where $\mathbf{k} \cdot \mathbf{B}_0 = 0$
  - No stabilizing effect of the magnetic field
  - The mode amplitude is constant along each field line
- Bending of field lines provides a restoring force and resists growth of the perturbation
- Magnetic shear limits the region of instability

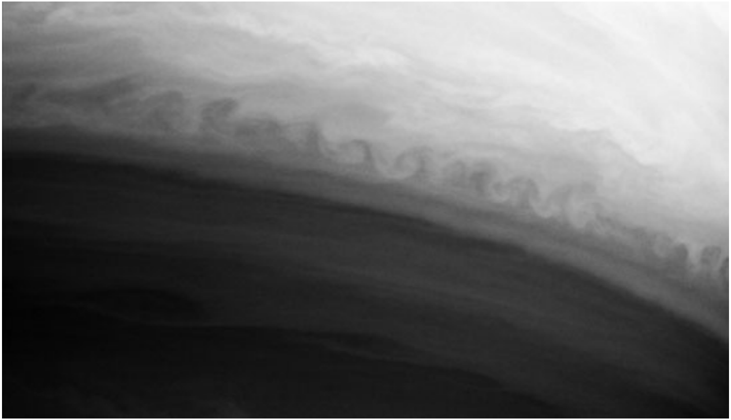
What's so important about $\mathbf{k} \cdot \mathbf{B}_0 = 0$?

- **Resonant surfaces** are locations where

$$\mathbf{k} \cdot \mathbf{B}_0 = 0 \tag{57}$$

- Instabilities are often localized to these surfaces where the magnetic field and the perturbation have the same pitch.
- When $\mathbf{k} \cdot \mathbf{B} \neq 0$, there is a compressional component to the mode. Compression of the magnetic field is an energy sink, so this effect is stabilizing.
- When $\mathbf{k} \cdot \mathbf{B} = 0$, the magnetic field cannot be compressed so the energy sink is eliminated.

The Kelvin-Helmholtz instability results from velocity shear



- Results in characteristic Kelvin-Helmholtz vortices
- Above: Kelvin-Helmholtz instability in Saturn's atmosphere

Overstable modes

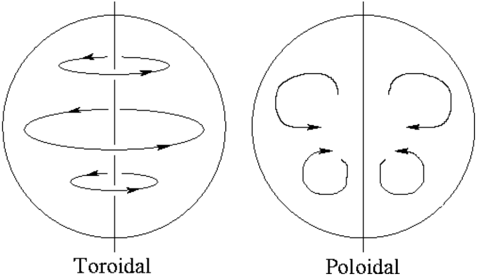
- I said that there are no overstable modes in MHD. Was I lying?
- Sort of! But we need to go beyond ideal MHD and include effects like
  - Radiative cooling
  - Anisotropic thermal conduction
- Examples include (e.g., Balbus & Reynolds 2010)
  - Magnetothermal instability
  - Heat flux buoyancy instability

  These are important in the intracluster medium of galaxy clusters.

Linear growth vs. nonlinear growth and saturation of instabilities

- In the initial linear stage, plasma instabilities grow exponentially
- This exponential growth continues until the linearization approximation breaks down
- Nonlinear growth is usually slower than exponential
- The saturation of instabilities helps determine the resulting configuration of the system
- Numerical simulations are usually needed to investigate nonlinear growth and saturation

Poloidal and toroidal mode numbers



- Instabilities in toroidal laboratory plasma devices are typically described using:
  - The poloidal mode number, m
  - The toroidal mode number, n
- Instabilities in a torus are naturally periodic, and (m, n) describe how many times the instability wraps around in each of the poloidal and toroidal directions
- Usually not important in astrophysics (except perhaps in global instabilities in accretion disks) but worth mentioning!

Summary

- There are several different formulations for gauging MHD stability
- The initial value formulation provides the most information but typically requires the most effort
- The normal mode formulation is more amenable to linear analysis but not nonlinear analysis
- The variational approach retains information about stability and provides limits on the growth rate
- The energy principle is the most elegant method for determining stability and provides insight into the mechanisms behind instabilities, but does not help us determine the growth rate