MHD Shocks and Discontinuities

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These lecture notes are based off of §5.6 of *The Physics of Plasmas* by Boyd and Sanderson; Chapter 5 of *Magnetohydrodynamics of the Sun* by Priest; §4.5 of Goedbloed & Poedts; Chapter 20 of Goedbloed, Keppens, & Poedts; lecture notes by Steve Cranmer; discussions with John Raymond, Mike Stevens, and Kelly Korreck; the review by Ghavamian et al. (2015); and lecture/presentation slides by Yosuke Mizuno, Dejan Urošević, Merav Opher, Anatoly Spitkovsky, Tom Hartquist, Nora Elisa Chisari, and others.

Outline

- Hydrodynamic shocks
- Magnetohydrodynamic shocks
  - Parallel shocks
  - Perpendicular shocks
  - Oblique shocks
- Shock thickness
- MHD discontinuities
  - Tangential discontinuity
  - Rotational discontinuity
- Applications
  - Supernova remnants
  - Laboratory experiments
  - Near-Earth space plasmas

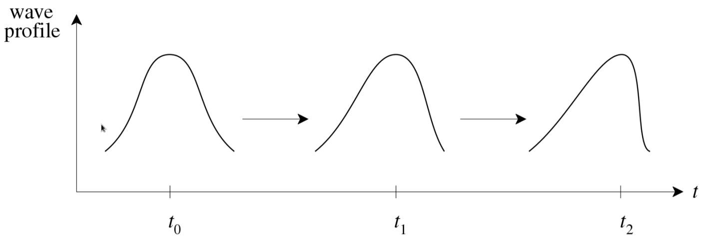
Shocks are important astrophysical phenomena

- Supernova remnant shock waves
  - Deposit energy and high metallicity plasma into ISM
  - Accelerate cosmic rays
  - Instigate star formation
- Heliospheric shocks
  - Frequently driven by solar eruptions
  - Routinely observed at 1 AU
  - The solar wind drives Earth's bow shock and the heliospheric termination shock
- Accretion shocks in clusters of galaxies
- Shocks in star formation regions
- Gamma ray bursts, supernovae, AGN jets, pulsar wind nebulae, etc.

Definition of a shock

- A shock is a discontinuity separating two different regimes in an otherwise continuous medium
- Shocks form when the velocities exceed the signal propagation speed
- In hydrodynamics, the only wave speed is the sound speed, c_s
- In MHD, there are different characteristic speeds for each of the three modes
- Not all discontinuities are shocks
  - Example: a static ideal gas where the pressure is uniform but there is a sharp transition in density and temperature

Nonlinear steepening of longitudinal waves

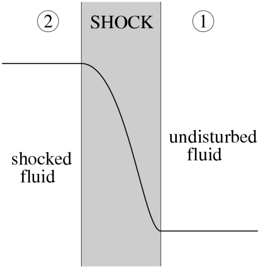


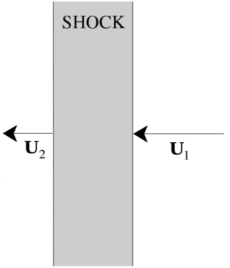
- When the amplitude of a disturbance is small enough for linear theory to apply, then it propagates as a wave
- If the disturbance becomes large enough, then nonlinear terms become important
- The crest of a wave propagates faster than the leading or trailing edge
  - The sound speed $c_s \equiv \sqrt{\frac{\gamma p}{\rho}}$ is greater at the crest
- The wave front steepens and a shock may form

Strategy for finding the jump conditions across the shock

- Choose a reference frame that is co-moving with the shock
- Put the equations of HD or MHD in integral form
  - Assume no time dependence
- Evaluate these integrals to find quantities that must be conserved across the shock
  - For example, matching the mass flux through the shock

Choose a reference frame that is co-moving with the shock





- Region 1 is the undisturbed fluid in front of the shock
- Region 2 as the disturbed fluid behind the shock
- Define $\mathbf{U}$ as the velocity field in the reference frame where the shock is stationary. If $\mathbf{V}_s$ is the velocity of the shock, then

$$\mathbf{U} \equiv \mathbf{V} - \mathbf{V}_s \tag{1}$$

- Define $\hat{\mathbf{x}}$ as the shock propagation direction

Intuitively finding a relation for conservation of mass

- The mass flux approaching the shock is $\rho_1 U_1$
- The mass flux leaving the shock is $\rho_2 U_2$
- In a steady state, these quantities must be equal
- The jump condition for mass continuity must then be

$$\rho_1 U_1 = \rho_2 U_2 \tag{2}$$

Put the continuity equation in integral form

- The continuity equation in conservative form is

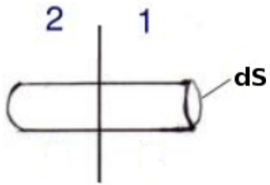
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{3}$$

- Set the time derivative equal to zero and use the divergence theorem to put the continuity equation in integral form

$$\oint_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot (\rho \mathbf{V}) = 0 \tag{4}$$

where $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\, \mathrm{d}S$

Using integrals to find a relation for conservation of mass



- Choose a cylinder extending between regions 1 & 2, as above
- There is no flux through the surface parallel to the flow
- The only flux is through the ends:

$$\oint \mathrm{d}\mathbf{S} \cdot (\rho \mathbf{U}) = -\int_1 \rho_1 U_1 \mathrm{d}S + \int_2 \rho_2 U_2 \mathrm{d}S = 0 \qquad (5)$$

$$\Rightarrow \rho_1 U_1 = \rho_2 U_2 \qquad (6)$$

Finding a jump condition for conservation of momentum

- The steady state momentum equation for HD in integral form is
$$\oint \mathrm{d}\mathbf{S} \cdot (\rho \mathbf{V}\mathbf{V} + p\mathbf{I}) = 0 \tag{7}$$
where $\mathbf{I}$ is the identity dyadic tensor.
- Evaluating this integral over the same cylinder with $\mathbf{B} = 0$ and velocity entirely in the x direction yields
$$\rho_1 U_1^2 + p_1 = \rho_2 U_2 + p_2 \tag{8}$$
- Recall that $\rho_1 U_1$ is the momentum density which is multiplied by U_1 again to become a momentum flux

Notation and definitions

- There are several standard ways to write jump conditions across a shock:

$$
\begin{aligned}
[\![\phi]\!] &= 0 \quad (9)\\
[\phi]_1^2 &= 0 \quad (10)\\
[\phi] &= 0 \quad (11)\\
\phi_2 - \phi_1 &= 0 \quad (12)
\end{aligned}
$$

  All of these notations are equivalent.
- Define the Mach number as

$$M \equiv \frac{U_1}{c_{s1}} \quad (13)$$

  where the upstream sound speed is $c_{s1} \equiv \sqrt{\frac{\gamma p_1}{\rho_1}}$.

The Rankine-Hugoniot jump conditions for a hydrodynamic shock

- Conservation of mass yields

$$[\![\rho U_x]\!] = 0 \tag{14}$$

- Conservation of momentum yields

$$\left[\!\!\left[\rho U_x^2 + p\right]\!\!\right] = 0 \tag{15}$$

- Conservation of energy yields

$$\left[\!\!\left[\rho \mathcal{I} U_x + p U_x + \frac{\rho U_x^3}{2}\right]\!\!\right] = 0 \tag{16}$$

where the internal energy density is

$$\mathcal{I} \equiv \frac{p}{(\gamma - 1)\,\rho} \tag{17}$$

Properties of hydrodynamic shocks

- The density ratio across the shock is

$$r \equiv \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \tag{18}$$

- The velocity ratio is

$$\frac{U_2}{U_1} = \frac{\rho_1}{\rho_2} = r^{-1} \tag{19}$$

- The pressure ratio is

$$R \equiv \frac{p_2}{p_1} = \frac{2\gamma M^2 - (\gamma-1)}{\gamma+1} \tag{20}$$

- The shock speed must exceed the sound speed ahead of the shock: $M \geq 1$
- Shocks are compressive: $\rho_2 \geq \rho_1$ and $p_2 \geq p_1$
- Shocks must increase entropy

What happens in the hypersonic limit?

- The hypersonic limit is when $M \to \infty$
- The density ratio becomes

$$r = \frac{\gamma + 1}{\gamma - 1} \tag{21}$$

- If $\gamma = \frac{5}{3}$, then the pressure increases without limit but density is only enhanced by a factor of 4
- In the isothermal limit of $\gamma \to 1$, the density ratio may become huge
    - Effective radiative cooling and thermal conduction allow greater density ratios

From HD to MHD

- Conservation of mass relation is unchanged
- Conservation of momentum must include the Lorentz force
- Conservation of energy must include magnetic energy
- Need additional constraints related to:
  - Divergence constraint
  - Conservation of magnetic flux

Modifying the conservation of momentum relation

- The steady state momentum equation for MHD is

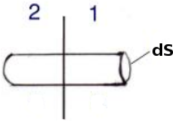
$$\oint \mathrm{d}\mathbf{S} \cdot \left[\rho \mathbf{V}\mathbf{V} + \left(p + \frac{B^2}{8\pi}\right)\mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi}\right] = 0 \tag{22}$$

  which includes contributions from the Maxwell and Reynolds stresses in the form of dyadic tensors.

- The integration process yields the corresponding jump condition

$$\left[\!\left[\rho \mathbf{U}\left(\mathbf{U} \cdot \hat{\mathbf{n}}\right) + \left(p + \frac{B^2}{8\pi}\right)\hat{\mathbf{n}} - \frac{\left(\mathbf{B} \cdot \hat{\mathbf{n}}\right)\mathbf{B}}{4\pi}\right]\!\right] = 0 \tag{23}$$

The divergence constraint



- The divergence constraint in integral form is

$$\oint_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \mathbf{B} = 0 \tag{24}$$

- Perform this integral over the same cylinder as before
- The contributions from the side parallel to the flow cancel out due to symmetry, so only the end contributions remain:

$$\int_1 \mathbf{B} \cdot \hat{\mathbf{n}}\, \mathrm{d}S + \int_2 \mathbf{B} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = 0 \tag{25}$$

$$\Rightarrow [\![B_x]\!] = 0 \tag{26}$$

The normal component of **B** remains constant across a shock

Continuity of the tangential component of $\mathbf{E}$

- From Faraday's law for a steady state, we need

$$\nabla \times \mathbf{E} = 0 \tag{27}$$

- The ideal Ohm's law in the shock frame is

$$\mathbf{E} + \frac{\mathbf{U} \times \mathbf{B}}{c} = 0 \tag{28}$$

- Integrate Eq. 27 over a thin strip including the shock, and use Stokes' theorem to convert it to a line integral
- The component of $\mathbf{E}$ tangential to the shock must be continuous

$$[\![\hat{\mathbf{n}} \times (\mathbf{U} \times \mathbf{B})]\!] = 0 \tag{29}$$

  This jump condition accounts for conservation of flux

Jump conditions for MHD shocks - Part 1

- Conservation of mass yields

$$[\![\rho \mathbf{U} \cdot \hat{\mathbf{n}}]\!] = 0 \tag{30}$$

- Conservation of momentum yields

$$\left[\!\!\left[\rho \mathbf{U} \left(\mathbf{U} \cdot \hat{\mathbf{n}}\right) + \left(p + \frac{B^2}{8\pi}\right) \hat{\mathbf{n}} - \frac{\left(\mathbf{B} \cdot \hat{\mathbf{n}}\right) \mathbf{B}}{4\pi}\right]\!\!\right] = 0 \tag{31}$$

- Conservation of energy yields

$$\left[\!\!\left[\mathbf{U} \cdot \hat{\mathbf{n}} \left\{ \left(\rho \mathcal{I} + \frac{\rho U^2}{2} + \frac{B^2}{8\pi}\right) + \left(p + \frac{B^2}{8\pi}\right) \right\} - \frac{\left(\mathbf{B} \cdot \hat{\mathbf{n}}\right) \left(\mathbf{B} \cdot \mathbf{U}\right)}{4\pi}\right]\!\!\right] = 0 \tag{32}$$

Jump conditions for MHD shocks - Part 2

- Continuity of the normal component of **B** yields

$$[\![\mathbf{B} \cdot \hat{\mathbf{n}}]\!] = 0 \tag{33}$$

- Continuity of the tangential component of **E** yields

$$[\![\hat{\mathbf{n}} \times (\mathbf{U} \times \mathbf{B})]\!] = 0 \tag{34}$$

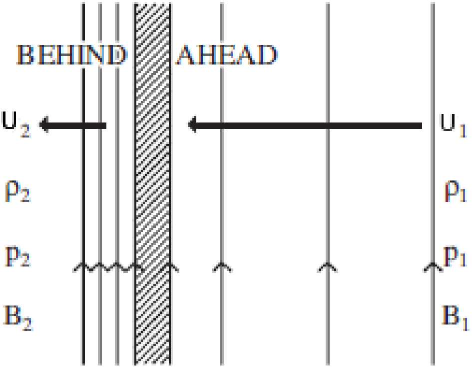
Examples

- The jump conditions contain rich physics related to different wave modes in MHD
- Define θ as the angle between $\mathbf{B}_1$ and $\hat{\mathbf{n}}$
  - $\theta = 0$ corresponds to a parallel shock
  - $\theta = \frac{\pi}{2}$ corresponds to a perpendicular shock
  - $0 < \theta < \frac{\pi}{2}$ corresponds to an oblique shock
- The jump conditions allow three types of discontinuities that are not shocks:
  - Contact discontinuity
  - Tangential discontinuity
  - Rotational discontinuity

Parallel shocks ($\theta = 0$)

- Here, $\theta = 0$ and both $\mathbf{U}_1$ and $\mathbf{B}_1$ are parallel to $\hat{\mathbf{n}}$
- Often favorable to particle acceleration (e.g., SN 1006)
- Simplest case: the magnetic field is parallel to the shock velocity and constant in front of and behind the shock ($\mathbf{B}_1 = \mathbf{B}_2$)
  - $\mathbf{B}$ drops out of the jump conditions
  - This solution reduces to a hydrodynamic shock
- Other possibility: a switch-on shock where $|B_2| > |B_1|$
  - The normal component of $\mathbf{B}$ is conserved ($[\![\mathbf{B} \cdot \hat{\mathbf{n}}]\!] = 0$)
  - $\mathbf{B}_2$ has a tangential component
  - Some of the flow energy is converted into magnetic energy

Perpendicular shocks ($\theta = \frac{\pi}{2}$)



- The upstream and downstream magnetic fields are both tangential to the shock front: $\mathbf{B}_1 = B_1\hat{\mathbf{y}}$ and $\mathbf{B}_2 = B_2\hat{\mathbf{y}}$
- Flow energy is converted to magnetic energy and heat

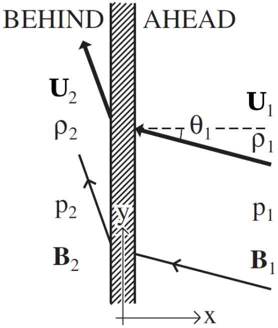
Perpendicular shocks ($\theta = \frac{\pi}{2}$)

- The shock speed must exceed the fast magnetosonic speed ahead of the shock:

$$U_1^2 > V_A^2 + c_s^2 \tag{35}$$

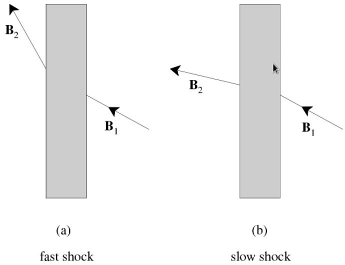
- The magnetic field is constant in direction and increases in magnitude by the same ratio as the density
- Perpendicular shocks are fast mode shocks
- There is no perpendicular shock corresponding to the slow wave because it does not propagate orthogonal to **B**

Oblique shocks ($0 < \theta < \frac{\pi}{2}$)



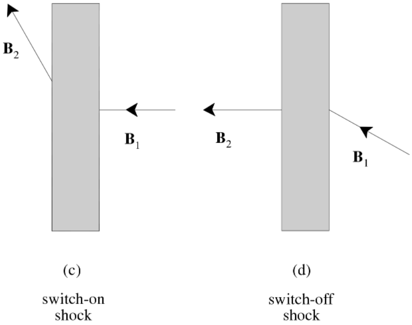
- Both **U** and **B** may change direction across the shock
- Must account for the tension force acting across a plane
- The jump conditions are simplified by choosing the de Hoffmann-Teller frame with $\mathbf{U}_1 \times \mathbf{B}_1 = 0$

Fast mode shocks and slow mode shocks are analogs of fast mode and slow mode waves



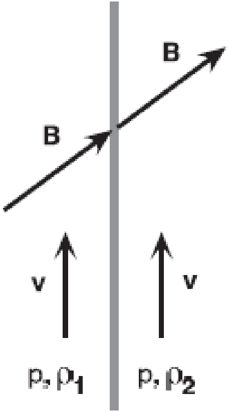
- Fast shocks increase the tangential component of **B**
  - Magnetic field is refracted away from the shock normal
  - The total magnetic field strength increases
- Slow shocks decrease the tangential component of **B**
  - Magnetic field is refracted toward the shock normal
  - The total magnetic field strength decreases
- Each type of shock must exceed the characteristic velocity of the corresponding wave mode

Switch-on shocks and switch-off shocks



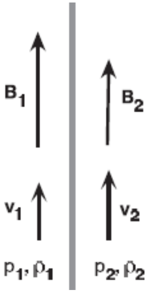
- Switch-on shocks
  - Tangential **B** develops as a result of the shock
  - Example of a fast mode shock
- Switch-off shocks
  - Tangential **B** disappears as a result of the shock
  - Example of a slow mode shock
- Remember: $[\![\mathbf{B} \cdot \hat{\mathbf{n}}]\!] = 0$! Only the tangential component of **B** may change.

Contact discontinuity



- All quantities except the density (and therefore temperature) are continuous across a contact discontinuity
- No flow across the discontinuity
- Because $[\![T]\!] \neq 0$, fast parallel thermal conduction will not let a contact discontinuity last long

Tangential discontinuity

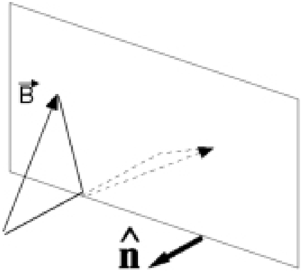


- Both **B** and **U** are tangential to the discontinuity
- No flow across the discontinuity
- The total pressure must be continuous across the discontinuity

$$\left[\!\left[p + \frac{B^2}{8\pi} \right]\!\right] = 0 \tag{36}$$

- In the perpendicular limit ($\theta \to \frac{\pi}{2}$), the slow shock reduces to a tangential discontinuity

Rotational discontinuity



- The magnetic field and plasma flow change direction but not magnitude
- There is mass flow across the discontinuity but both the density and normal component of velocity are constant: $[\![\rho]\!] = 0$ and $[\![U_x]\!] = 0$
- Incompressible like a shear Alfvén wave
- Not a shock; and $M_1 = M_2 = 1$
- Frequently observed in the fast solar wind

What are the limits of this analysis?

- The ideal MHD approximation breaks down near the discontinuity
  - Need kinetic theory
- May need to consider effects such as
  - Radiative cooling
  - Partial ionization
  - Thermal conduction
- Shocks may propagate into a non-uniform or clumpy medium and/or
- Particle acceleration & energetic particles not considered
- Shocks may give rise to various instabilities
  - Weibel instability
  - Streaming instabilities
  - Bell's instability
  - Rayleigh-Taylor

What sets the shock thickness?

- The shock thickness δ depends on the dominant dissipation mechanism(s)
- Order of magnitude estimate for kinetic viscosity
  - The viscosity ν is often given in units of $\frac{\text{length}^2}{\text{time}}$
  - Dimensional analysis yields

$$\delta \sim \frac{\nu}{U_1} \tag{37}$$

    The thickness is the scale that yields $\mathrm{Re} \sim 1$
  - From kinetic theory, this yields δ of order an ion mean free path

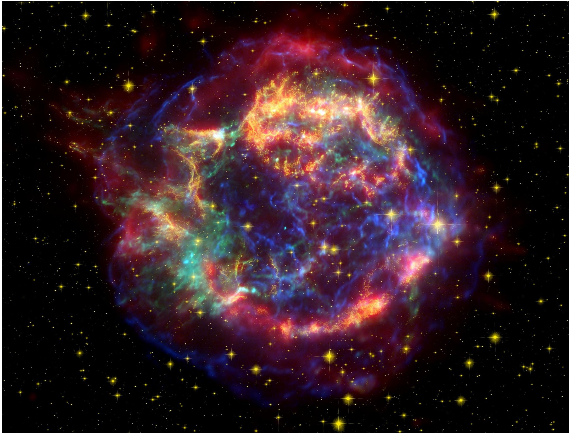
What governs the structure of collisionless shocks?

- Shock thickness depends on β, θ, M, type of shock, and other parameters
- Laminar collisionless shocks often have a thickness of order an ion inertial length: $d_i = c/\omega_{pi}$
  - There will typically be structure on multiple scales [e.g., the ion and electron Larmor radii (r_{Li}, r_{Le}), and the electron inertial length ($d_e = c/\omega_{pe}$)]
- Understanding the structure of collisionless shocks requires
  - Kinetic simulations/kinetic theory
  - Space & laboratory observations
  - Astrophysical observations
- Energetic particles modify the shock structure by generating instabilities while getting bounced back and forth

Simulating shocks provides significant numerical challenges

- Numerical schemes with inadequate dissipation often lead to unphysical grid scale oscillations near steep gradients
  - Similar to Gibbs phenomena
- High order numerical methods are often reduced to first order accuracy near discontinuities
- Adding uniform viscosity smooths out low M shocks but is insufficient for high M shocks
- Modern schemes with improved accuracy for shock capturing
  - Total variation diminishing (TVD) schemes
  - Essentially Non-Oscillatory (ENO) schemes
  - Monotonic Upstream-centered Schemes for Conservation Laws (MUSCL)
  - Discontinuous Galerkin method

Application: Supernova remnant (SNR) shock waves

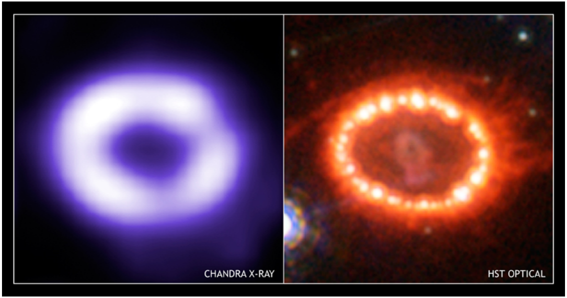


- Supernovae cause chemical enrichment, energy input, heating, turbulence, and particle acceleration in the ISM
- *Above:* A composite of multiwavelength observations of Cassiopeia A (red: infrared from Spitzer; orange: visible from HST, and blue/green: X-rays from Chandra)

How do supernova remnant shock waves fit in with the rest of astrophysics?

- Transport of enriched mass from supernovae into the ISM
- Energy input into ISM
- Acceleration of cosmic rays (CRs)
  - Need $\sim$10% of shock energy to go into CRs
  - CRs increase pressure support of ISM
  - CRs cause slight ionization of gas in molecular clouds
- The physics of SNR shock waves has significant consequences on star formation and galactic structure/evolution

First phase: Free expansion



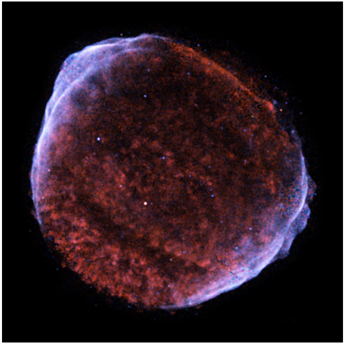
SN 1987A

- Mass of ejecta > swept-up mass
- Shock velocity remains roughly constant

$$R(t) \approx V_0 t \tag{38}$$

- Dynamics determined primarily by energy of explosion (V_0)
- Free expansion stage lasts hundreds of years

Second phase: Adiabatic expansion (Sedov-Taylor)



SN 1006

- Mass of ejecta $<$ swept-up mass
- Cooling time is typically longer than age of remnant
- Lasts tens of thousands of years
- *Above:* SN 1006. Red corresponds to X-ray emission from hot plasma and blue corresponds to X-ray emission from energetic electrons. Particle acceleration is seemingly most efficient when the shock is $\sim$parallel.

Using dimensional analysis to determine an expression for self-similar expansion

- Start from an expression that goes as

$$R(t) \sim E^{\alpha} \rho^{\beta} t^{\gamma} \tag{39}$$

where $R(t)$ is the radius of the remnant, E is the supernova energy, ρ is the density of the ambient ISM, and t is time.

- Using dimensional analysis, we find that

$$\alpha = \frac{2}{5}, \quad \beta = \frac{1}{5}, \quad \gamma = -\frac{1}{5} \tag{40}$$

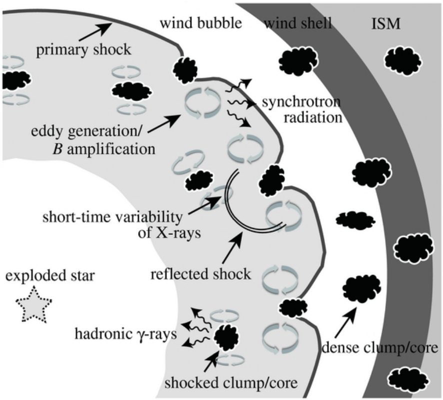
- The Sedov solution for self-similar expansion is

$$R(t) \sim \left(\frac{E}{\rho} \right)^{1/5} t^{2/5} \tag{41}$$

Third phase: radiative (snowplow)

- Mass of ejecta $\ll$ swept-up mass
- The expansion slows
- Radiative cooling is important energy loss mechanism
  - The shock becomes 'isothermal'
  - Allows a high compression ratio
- Lasts hundreds of thousands of years
- Hot diffuse plasma inside remnant has not had time to cool
- Expansion velocity eventually becomes comparable to characteristic ISM velocities $\rightarrow$ remnant fades away
  - The shock eventually becomes a wave

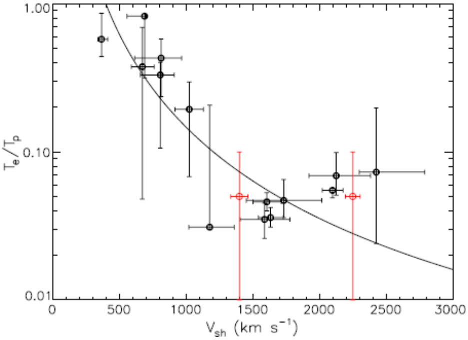
The structure of the ambient medium will normally be inhomogeneous



- Instabilities will also lead to structure in the shock and ejecta

Kinetic processes govern the partition between electron and ion heating, as well as equilibration between species

Fig. 2 The correlation between β ($\equiv T_e/T_p$) and shock speed for Balmer-dominated SNR shocks is shown, using results from van Adelsberg et al. (2008). The *solid curve* shows the V_S^{-2} dependence inferred by Ghavamian et al. (2007) for Balmer-dominated SNR shocks. New results quoted from the NE and NW portions of Tycho's SNR (this paper) are marked in *red*. The apparent upturn at $V_s \gtrsim 2000$ km s^{-1} and its associated error bars are discussed more fully in the text

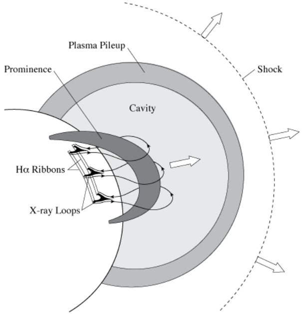


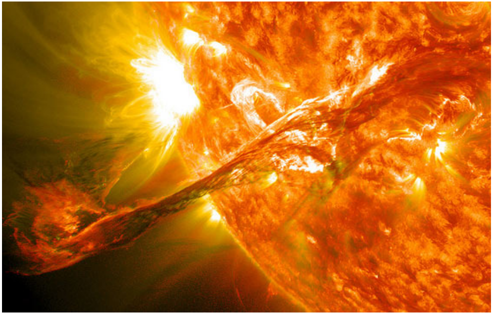
- The jump conditions include the total pressure, but do not say anything about the partition between p_e and p_i
- More effective electon heating in slower shocks
- Slow temperature equilibration between ions and electrons will occur due to Coulomb collisions
    - Is quicker equilibration possible due to collisionless plasma processes?

Open questions on supernova remnant shock waves

- How are cosmic rays accelerated?
- What instabilities arise near the shock?
- What processes govern the magnetic field strength and structure near the shock?
- How is energy partitioned between ions, electons, and cosmic rays?
- How is energy transferred between ions and electrons behind the shock?

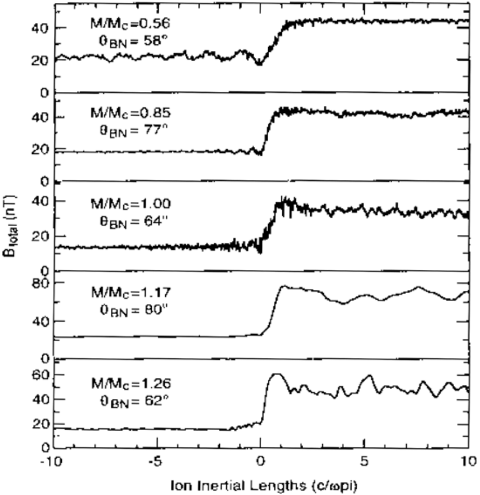
Shocks driven by solar flares and coronal mass ejections





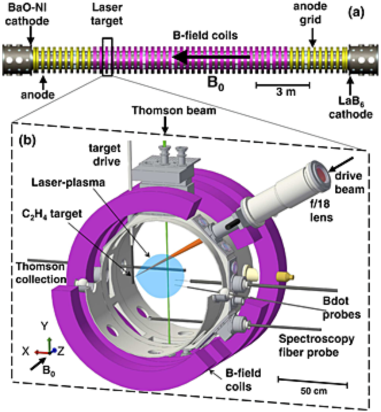
- The outward shocks precede the main ejecta and contribute to solar energetic particle (SEP) events
- Slow mode and fast mode shocks may develop below the prominence due to reconnection
- *Right:* An erupting prominence observed by SDO at 30.4 nm (He II Lyα; plasma at $\lesssim 10^5$ K)

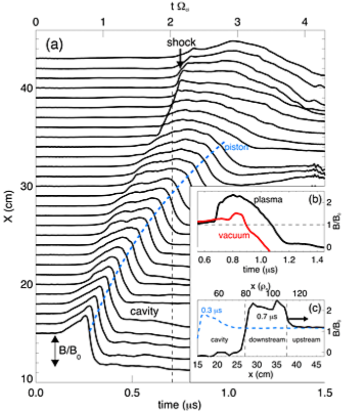
Collisionless shocks in near-Earth space plasmas



- In situ measurements by spacecraft in the magnetosphere and nearby solar wind allow detailed study of interplanetary shocks driven by solar eruptions and Earth's bow shock

Laser produced plasmas allow the study of astrophysically relevant collisionless shocks





- *Left:* Schematic of (a) Large Plasma Device and (b) the laser-target and diagnostics configuration.
- *Right:* (a) Magnetic stack plots of B_z as a function of time for various distances from the target. (b) Comparison of $B_z(t)$ at $x = 35$ cm with (black) and without (red) the ambient plasma. (c) Structure of the pulse before ($t = 0.3$ μs) and after a shock is formed ($t = 0.7$ μs).

Summary

- Shocks are discontinuities that separate two different regimes in an otherwise continuous medium
- The jump conditions across a hydrodynamic shock are described by the Rankine-Hugoniot relations
  - Conservation of mass, momentum, and energy
- In MHD shocks, the jump conditions account for **B** and **E**
  - The component of **B** normal to the shock remains constant
  - The component of **E** tangential to the shock remains constant
- The MHD jump conditions allow for shocks that correspond to different wave modes
- Contact, tangential, and rotational discontinuities are solutions that are not shocks
- Collisionless processes govern shocks in supernova remnants, solar/space plasmas, laser-produced plasmas, and elsewhere in astrophysics