The Virial Theorem, MHD Equilibria, and Force-Free Fields

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These lecture notes are largely based on *Plasma Physics for Astrophysics* by Russell Kulsrud, *Lectures in Magnetohydrodynamics* by the late Dalton Schnack, *Ideal Magnetohydrodynamics* by Jeffrey Freidberg, *Hydrodynamic and Hydromagnetic Stability* by S. Chandrasekhar, *Classical Electrodynamics* by J. Jackson, and examples by A. Savcheva and A. Spitkovsky.

Outline

- We will look at the properties and key characteristics of MHD equilibria. Our discussion will focus on:
  - The virial theorem

$$0 = 2\mathcal{E}_V + 3\left(\gamma - 1\right)\mathcal{E}_p + \mathcal{E}_B + \mathcal{E}_g$$

  - MHD equilibria

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \nabla p$$

  - Force-free fields

$$\mathbf{J} \times \mathbf{B} = 0 \Rightarrow \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

The Virial Theorem for MHD (following Kulsrud §4.6)

- The Virial Theorem allows us to understand broadly the equilibrium properties of a system in terms of energies
- Suppose there exists a magnetized plasma within a finite volume. The scalar moment of inertia is

$$\mathcal{I} = \frac{1}{2} \int_{\mathcal{V}} \rho r^2 \, \mathrm{d}\mathcal{V} \tag{1}$$

  where r is the position vector about some arbitrary origin
- Our strategy:
  - Calculate $\mathrm{d}\mathcal{I}/\mathrm{d}t$ and $\mathrm{d}^2\mathcal{I}/\mathrm{d}t^2$
  - Ignore surface integrals by assuming the volume is large
  - Put the result in terms of energies
  - Set $\mathrm{d}^2\mathcal{I}/\mathrm{d}t^2 = 0$ for an equilibrium
  - Determine the conditions under which the resulting equation can be satisfied

Take the first time derivative of $\mathcal{I}$

- Use the continuity equation, the radial form $\nabla r^2 = 2\mathbf{r}$, the identity $\nabla \cdot (f\mathbf{A}) \equiv f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$, and Gauss' theorem.

$$\begin{aligned} \frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} &= \frac{1}{2}\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} r^2 \,\mathrm{d}\mathcal{V} = -\frac{1}{2}\int_{\mathcal{V}} \nabla \cdot (\rho \mathbf{V})\, r^2 \,\mathrm{d}\mathcal{V} \\ &= -\frac{1}{2}\int_{\mathcal{V}} \nabla \cdot \left(\rho \mathbf{V} r^2\right) \mathrm{d}\mathcal{V} + \frac{1}{2}\int \rho \mathbf{V} \cdot \nabla r^2 \,\mathrm{d}\mathcal{V} \\ &= -\frac{1}{2}\oint_{\mathcal{S}} r^2 \rho \mathbf{V} \cdot \mathrm{d}\mathbf{S} + \int_{\mathcal{V}} \rho \mathbf{V} \cdot \mathbf{r} \,\mathrm{d}\mathcal{V} \end{aligned} \qquad (2)$$

- The moment of inertia changes when mass enters or leaves the system, or mass moves toward or away from the origin
- Consider a volume large enough so that no mass enters or leaves. The surface integral then vanishes:

$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}t} = \int_{\mathcal{V}} \rho \mathbf{V} \cdot \mathbf{r} \,\mathrm{d}\mathcal{V} \qquad (3)$$

Take the second time derivative of $\mathcal{I}$

- Use the momentum equation, $\nabla \mathbf{r} = \mathbf{I}$, the tensor identity $\mathbf{A} \cdot \nabla \cdot \mathbf{T} = \nabla \cdot (\mathbf{A} \cdot \mathbf{T}) + \mathbf{T} : \nabla \mathbf{A}$, and Gauss' theorem

$$\begin{aligned} \frac{\mathrm{d}^2 \mathcal{I}}{\mathrm{d}t^2} &= \int_{\mathcal{V}} \mathbf{r} \cdot \frac{\partial}{\partial t} (\rho \mathbf{V}) \, \mathrm{d}\mathcal{V} = -\int_{\mathcal{V}} (\nabla \cdot \mathbf{T}) \cdot \mathbf{r} \, \mathrm{d}\mathcal{V} \\ &= -\int_{\mathcal{V}} \nabla \cdot (\mathbf{T} \cdot \mathbf{r}) \, \mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \mathbf{T} : \nabla \mathbf{r} \, \mathrm{d}\mathcal{V} \\ &= -\oint_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \mathbf{T} \cdot \mathbf{r} + \int_{\mathcal{V}} \mathrm{trace}\,(\mathbf{T}) \, \mathrm{d}\mathcal{V} \end{aligned} \tag{4}$$

The first term represents surface stresses. If we assume that surface stresses are negligible, then we are left with

$$\frac{\mathrm{d}^2 \mathcal{I}}{\mathrm{d}t^2} = \int_{\mathcal{V}} \mathrm{trace}\,(\mathbf{T}) \, \mathrm{d}\mathcal{V} \tag{5}$$

Let's look again at the stress tensor $\mathbf{T}$ (with gravity)

- We need to take

$$\mathbf{T} = \rho\mathbf{V}\mathbf{V} + p\mathbf{I} + \frac{B^2}{8\pi}\mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} + \frac{(\nabla\phi)^2\,\mathbf{I}}{8\pi G} - \frac{(\nabla\phi)(\nabla\phi)}{4\pi G} \tag{6}$$

- The Reynolds stress is

$$\rho\mathbf{V}\mathbf{V} = \begin{pmatrix} V_x V_x & V_x V_y & V_x V_z \\ V_x V_y & V_y V_y & V_y V_z \\ V_x V_z & V_y V_z & V_z V_z \end{pmatrix} \tag{7}$$

Then take its trace by adding up the diagonal elements:

$$\begin{aligned} \mathrm{trace}\,(\rho\mathbf{V}\mathbf{V}) &= \rho\left(V_x^2 + V_y^2 + V_z^2\right) \\ &= \rho V^2 \end{aligned} \tag{8}$$

It's just twice the kinetic energy density!

Now evaluate the traces of the other terms in $\mathbf{T}$

- The traces of the stress tensors yield energy densities times constants!

$$\mathrm{trace}\,(p\mathbf{I}) = 3p \tag{9}$$

$$\mathrm{trace}\left(\frac{B^2}{8\pi}\mathbf{I} - \frac{\mathbf{BB}}{4\pi}\right) = \frac{B^2}{8\pi} \tag{10}$$

$$\mathrm{trace}\left(\frac{(\nabla\phi)^2\,\mathbf{I}}{8\pi G} - \frac{(\nabla\phi)(\nabla\phi)}{4\pi G}\right) = -\frac{(\nabla\phi)^2}{8\pi G} \tag{11}$$

- Recall that the internal energy density is given by $p/(\gamma - 1)$

Now let's put these back into the volume integral

- By replacing $\mathrm{trace}\,(\mathbf{T})$ in Eq. 5 we arrive at

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}t^2} &= \int_{\mathcal{V}} \left(\rho V^2 + 3p + \frac{B^2}{8\pi} - \frac{(\nabla\phi)^2}{8\pi G}\right) \mathrm{d}\mathcal{V} \\ &= 2\mathcal{E}_V + 3\,(\gamma - 1)\,\mathcal{E}_p + \mathcal{E}_B + \mathcal{E}_g \end{aligned} \tag{12}$$

  where

  - Kinetic energy: $\mathcal{E}_V \geq 0$
  - Internal energy: $\mathcal{E}_p \geq 0$
  - Magnetic energy: $\mathcal{E}_B \geq 0$
  - Gravitational energy: $\mathcal{E}_g \leq 0$ (only possible negative term!)
- In an equilibrium, $\frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}t^2}$ must equal zero.
- The Virial Theorem is:

$$0 = 2\mathcal{E}_V + 3\,(\gamma - 1)\,\mathcal{E}_p + \mathcal{E}_B + \mathcal{E}_g \tag{13}$$

What happens when we neglect magnetic and internal energy?

- If $\mathcal{E}_B = \mathcal{E}_p = 0$, then we recover

$$\mathcal{E}_V = -\frac{1}{2}\mathcal{E}_g \tag{14}$$

  The kinetic energy must equal half the magnitude of the gravitational energy.
- This is a well-known result in self-gravitating systems such as star clusters, galaxies, and galaxy clusters
- This result has been used to infer the presence of dark matter

What happens when we drop gravity in a static system?

- In the absence of gravity and bulk motions, we are left with

$$0 = 3\,(\gamma - 1)\,\mathcal{E}_p + \mathcal{E}_B \tag{15}$$

  But $\mathcal{E}_p \geq 0$ and $\mathcal{E}_B \geq 0$! We have a contradiction!
- **A magnetized plasma cannot be in MHD equilibrium under forces generated solely by its own internal currents.**
- Equilibria are possible if there are external currents as in laboratory plasmas
  - Accounted for from the surface integrals we dropped
- In astrophysics, this might not be satisfied[1]

[1] One might say, neglecting gravity will be your downfall!

What limits on the magnetic energy does the Virial Theorem imply?

- If $\mathcal{E}_V = \mathcal{E}_p = 0$, then

$$\mathcal{E}_B + \mathcal{E}_g = 0 \tag{16}$$

  For a stable equilibrium, the magnetic energy must not exceed the magnitude of the gravitational energy.
- The virial theorem provides broad insight into the equilibrium properties of a relaxed system without having to worry about the details

MHD Equilibria

- We often care about systems that are in equilibrium
- Let's look at the momentum equation (neglecting gravity)

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p \tag{17}$$

- For a static equilibrium, the configuration must have

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \nabla p \tag{18}$$

in the absence of other forces

Properties of MHD equilibria

- Dot $\mathbf{B}$ with the equilibrium equation:

$$\begin{aligned} \mathbf{B} \cdot (\nabla p) &= \mathbf{B} \cdot \left(\frac{\mathbf{J} \times \mathbf{B}}{c} \right) && (19) \\ \mathbf{B} \cdot \nabla p &= 0 && (20) \end{aligned}$$

  where we use that $\mathbf{B}$ is orthogonal to $\mathbf{J} \times \mathbf{B}$.
  - $\mathbf{B} \cdot \nabla p$ is the directional derivative of p in the direction of $\mathbf{B}$
  - Plasma pressure is constant along magnetic field lines
- Similiarly, if we dot $\mathbf{J}$ with the equilibrium equation then

$$\mathbf{J} \cdot \nabla p = 0 \qquad (21)$$

  since $\mathbf{J}$ is orthogonal to $\mathbf{J} \times \mathbf{B}$ also.

Effects of fast thermal conduction on equilibria

- Ideal MHD does not include thermal conduction
- However, thermal conduction is very fast along field lines!
- If temperature is approximately constant along field lines, then

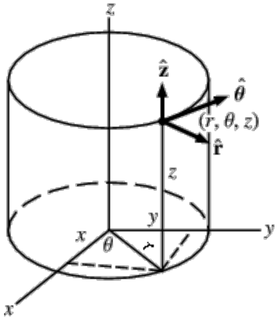
$$\mathbf{B} \cdot \nabla T \approx 0 \tag{22}$$

For $p = nkT$, then we also have

$$\mathbf{B} \cdot \nabla n \approx 0 \tag{23}$$

Eqs. 22 and 23 are not exact results, but rather commonly result from fast parallel thermal conduction.

We will next consider 1D and 2D equilibria



- We will use cylindrical coordinates (r,θ,z) such that

$$\mathbf{B}(r,z) = B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}} \qquad (24)$$

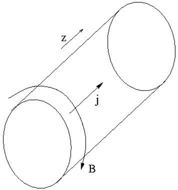
Example: equilibria with a unidirectional magnetic field

- Consider a configuration where the magnetic field is purely in the $\hat{\mathbf{z}}$ direction
- The equilibrium condition is then

$$p + \frac{B_z}{8\pi} = p_{tot} \tag{25}$$

  where the total pressure, p_{tot}, is a constant
- The tension forces disappear because the field lines are straight

Example: consider 1D cylindrical equilibria



- A 'Z-pinch' (above) has current flowing in the $\hat{\mathbf{z}}$ direction so that $\mathbf{B}$ is purely azimuthal
- A 'θ-pinch' has current flowing in the $\hat{\boldsymbol{\theta}}$ direction so $\mathbf{B}$ is purely axial
- A 'screw pinch' has components of $\mathbf{J}$ and $\mathbf{B}$ in both the axial and azimuthal directions
- For these configurations, we look for solutions of the form

$$p = p(r)\ ;\ \mathbf{J} = J_\theta(r)\hat{\boldsymbol{\theta}} + J_z(r)\hat{\mathbf{z}}\ ;\ \mathbf{B} = B_\theta(r)\hat{\boldsymbol{\theta}} + B_z(r)\hat{\mathbf{z}} \quad (26)$$

for which $\nabla \cdot \mathbf{B} = 0$ is trivially satisfied

Finding a Z-pinch 1D equilibrium

- Set $J_\theta = 0$ and $\mathbf{B}_z = 0$ since current is purely axial
- Ampere's law becomes

$$J_z(r) = \frac{c}{4\pi}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_\theta\right) \tag{27}$$

The $\hat{\mathbf{r}}$ component of the momentum equation is

$$J_z B_\theta = \frac{\mathrm{d}p}{\mathrm{d}r} \tag{28}$$

We then apply Eq. 27

$$\frac{\mathrm{d}p}{\mathrm{d}r} + \frac{c}{4\pi}\frac{B_\theta}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rB_\theta) = 0 \tag{29}$$

Finding a Z-pinch 1D equilibrium

- This can be rearranged to

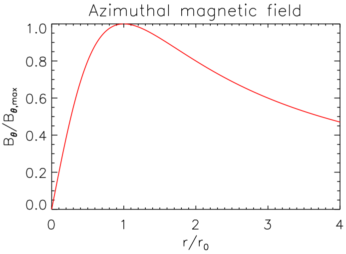
$$\frac{\mathrm{d}}{\mathrm{d}r}\underbrace{\left(p+\frac{B_\theta^2}{8\pi}\right)}_{\text{total pressure}}+\underbrace{\frac{B_\theta^2}{4\pi r}}_{\text{tension}}=0 \tag{30}$$

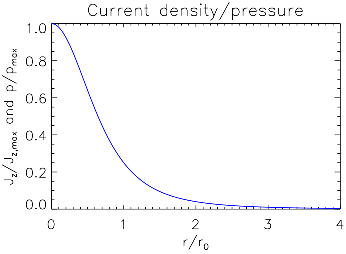
  Or, putting this in terms of the curvature vector,

$$\nabla_\perp\left(p+\frac{B^2}{8\pi}\right)-\frac{B^2}{4\pi}\boldsymbol{\kappa}=0 \tag{31}$$

- Total pressure and tension balance each other

A Z-pinch equilibrium can be found by specifying $B_\theta(r)$ and then solving for $p(r)$





- Shown above is the 'Bennett pinch' with

$$B_\theta \propto \frac{r}{r^2 + r_0^2} \quad ; \quad p, J_z \propto \frac{r_0^2}{(r^2 + r_0^2)^2} \tag{32}$$

- If the domain is $r \in [0, \infty]$ then the magnetic energy diverges! Need an outer wall, which is not present in astrophysics. Recall the Virial Theorem...

Axisymmetric equilibria are found by solving the *Grad-Shafranov equation*

- Fundamentally important for fusion devices like tokamaks
- Astrophysical applications include:
  - Magnetic flux ropes in the solar corona/wind and planetary magnetospheres
  - Compact object magnetospheres

The elements of the Grad-Shafranov equation

- The Grad-Shafranov equation comes from the equilibrium relation

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \nabla p \tag{33}$$

- We introduce a flux function ψ such that

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{34}$$

$$B_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{35}$$

  which satisfies the divergence constraint for any $B_\theta(r, z)$.

- Contours of constant ψ represent the projection of magnetic field lines into the poloidal (r-z) plane
- Both p and rB_θ are functions of ψ alone:

$$p = p(\psi) \tag{36}$$

$$rB_\theta = F(\psi) \tag{37}$$

How do we find a solution of the Grad-Shafranov equation?

- The Grad-Shafranov equation is given by

$$\Delta^* \psi + F \frac{\mathrm{d}F}{\mathrm{d}\psi} = -4\pi r^2 \frac{\mathrm{d}p}{\mathrm{d}\psi}, \tag{38}$$

  where

$$\Delta^* \equiv r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \tag{39}$$

$$F(\psi) \equiv r B_\theta \tag{40}$$

- To solve the Grad-Shafranov equation, we need to
  - Specify $p(\psi)$
  - Specify $F(\psi)$
  - Solve for ψ
- The Grad-Shafranov equation is usually solved numerically

Equilibria and the Virial Theorem

- The natural state of a flux rope is to try to expand to infinity
- In laboratory plasmas, conducting wall outer boundaries and externally applied magnetic fields prevent this
  - These show up as surface integrals in the Virial Theorem
- In astrophysical plasmas, a flux rope can be held in place from $\mathbf{J} \times \mathbf{B}$ and $-\nabla p$ forces from the surrounding medium
- The surrounding medium, in turn, can be held in place by gravitational forces
  - Example: the solar corona

How does gravity change things?

- The equilibrium condition becomes

$$0 = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p + \rho \mathbf{g} \qquad (41)$$

- **B** and **J** are no longer necessarily orthogonal to ∇p

$$\mathbf{B} \cdot \nabla p = \rho \mathbf{B} \cdot \mathbf{g} \qquad (42)$$

$$\mathbf{J} \cdot \nabla p = \rho \mathbf{J} \cdot \mathbf{g} \qquad (43)$$

- A radially outward flux tube reduces to the case of hydrostatic equilibrium: $\nabla p = \rho \mathbf{g}$
- The Grad-Shafranov equation can be generalized to include gravitational forces
  - I decided against assigning this as a homework problem.

Force-free fields

- When pressure is constant or in the limit of $\beta \to 0$, the pressure gradient force vanishes. Equilibria then have

$$\mathbf{J} \times \mathbf{B} = 0 \tag{44}$$

- Such configurations are called 'force-free' because there is no Lorentz force and no plasma pressure gradient force
- Using Ampere's law, this reduces to

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \tag{45}$$

  where α is constant along field lines
- Vector fields parallel to their own curl are called Beltrami fields

Linear force-free fields have constant α

- Start with the condition for a force-free field:

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \tag{46}$$

Take its curl and use that α is constant:

$$\nabla \times (\nabla \times \mathbf{B}) = \alpha \nabla \times \mathbf{B} \tag{47}$$

Use Eq. 46 for the RHS and vector identities

$$\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \alpha^2 \mathbf{B} \tag{48}$$

Linear force-free fields obey the Helmholtz equation

$$\nabla^2 \mathbf{B} + \alpha^2 \mathbf{B} = 0 \tag{49}$$

which can be solved using separation of variables, Green's functions, Fourier series, or numerically.

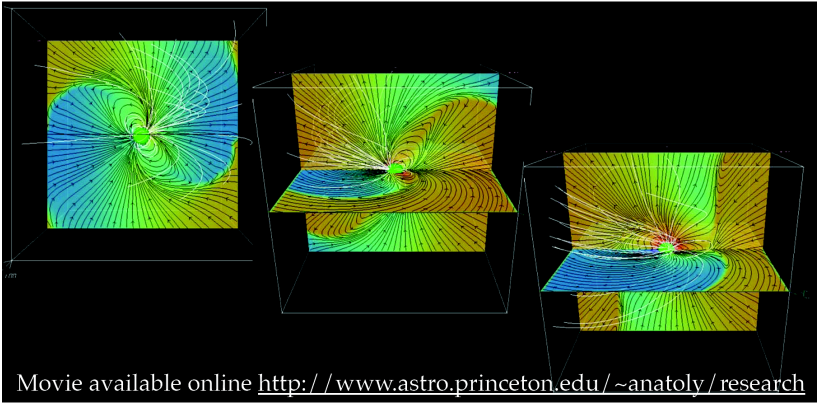
Nonlinear force-free fields (NLFFF) have non-constant α

- Finding nonlinear force-free fields requires solving

$$\nabla \times \mathbf{B} = \alpha(\mathbf{x})\,\mathbf{B} \tag{50}$$

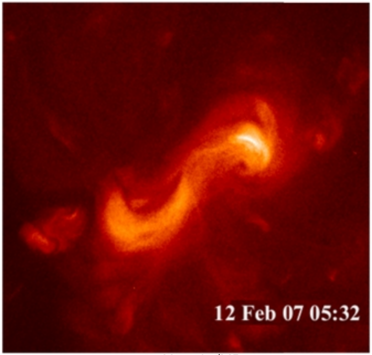
  where α must still remain constant along field lines
- Some analytical examples exist in 2D
- 3D solutions must generally be found numerically

Example: pulsar magnetospheres



- Force-free field models are able to capture features that do not exist in a vacuum solution
- Figure of oblique rotator by Anatoly Spitkovsky. Color marks direction of toroidal field.

Example: NLFFF modeling of sigmoidal solar active regions



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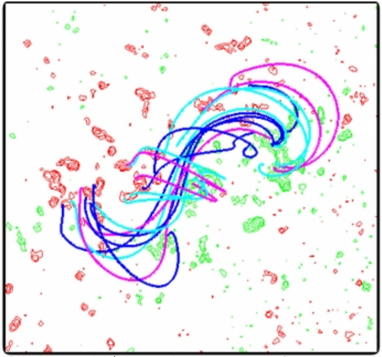

NLFFF model/photospheric magnetic flux contours

- Significant free energy is stored in the sheared magnetic field of sigmoids (S-shaped structures in the corona)
- The modeling strategy is to insert a flux rope into a potential field model, let it relax, and see if it matches the observed X-ray emission (Savcheva et al. 2012)

Potential magnetic fields represent the minimum energy condition for a configuration's boundary conditions

- Potential fields are force-free fields with $\alpha = 0$
- Potential fields have $\mathbf{J} = 0$, so Ampere's law becomes

$$0 = \nabla \times \mathbf{B} \tag{51}$$

The curl of a gradient is zero, so this is satisfied if

$$\mathbf{B} = -\nabla\zeta \tag{52}$$

where ζ is a scalar magnetic potential

- Taking the divergence of both sides shows that ζ can be found by solving Laplace's equation

$$\nabla \cdot \mathbf{B} = -\nabla \cdot \nabla\zeta \tag{53}$$

$$0 = \nabla^2\zeta \tag{54}$$

Potential magnetic fields may be calcuated using Laplace's equation for the vector potential

- From Ampere's law with no currents we know that

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (55)$$

$$0 = \frac{c}{4\pi}\nabla \times \mathbf{B} \qquad (56)$$

From these two expressions we arrive at

$$\nabla \times (\nabla \times \mathbf{A}) = 0 \qquad (57)$$

Using vector identities and choosing the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) we arrive at

$$\nabla^2 \mathbf{A} = 0 \qquad (58)$$

Determining the magnetic free energy of a system

- The magnetic free energy of a system is given by

$$\Delta\mathcal{E}_M \equiv \int_{\mathcal{V}} \left(\frac{B^2}{8\pi} - \frac{B_p^2}{8\pi} \right) \mathrm{d}\mathcal{V} \tag{59}$$

  where B_p is the potential field solution for the system's boundary conditions

- However, the quantity

$$\frac{B^2}{8\pi} - \frac{B_p^2}{8\pi} \tag{60}$$

  should *not* be considered a free energy density since free energy is an integral quantity

- Current density is a good proxy for how much stress exists in a magnetic field

Summary

- The Virial Theorem allows us to understand key requirements for equilibria even before we know the details
  - A plasma cannot be in an MHD equilibrium generated solely by its internal currents
- 2D MHD equilibria are found using the *Grad-Shafranov equation*
  - However, many equilibria are *unstable*
  - 3D MHD equilibria are difficult to find and/or generalize
- Force-free fields have $\mathbf{J} \times \mathbf{B} = 0$ so that $\nabla \times \mathbf{B} = \alpha \mathbf{B}$
  - Linear force-free fields have constant α and can be solved for using the Helmholtz equation
  - Nonlinear force-free fields have nonconstant α and usually are solved numerically
- Potential fields are the minimum energy state for given boundary conditions and are force-free fields with $\alpha = 0$