# Conservation Laws in Ideal MHD

Nick Murphy

Harvard-Smithsonian Center for Astrophysics

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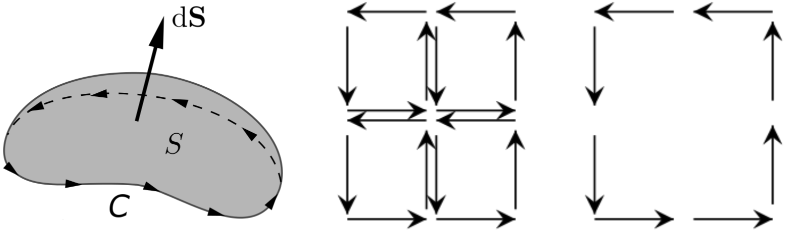
These lecture notes are largely based on *Plasma Physics for Astrophysics* by Russell Kulsrud, *Lectures in Magnetohydrodynamics* by the late Dalton Schnack, *Ideal Magnetohydrodynamics* by Jeffrey Freidberg, *Magnetic Reconnection* by Eric Priest and Terry Forbes, and various papers by M. Berger & J. Perez. The description of Stokes' Theorem and the Divergence Theorem come from Stewart's Multivariable Calculus with figures adapted from Wikimedia Commons and R. Fitzpatrick's online notes.

# Outline

- Mathematical interlude:
  - Stokes' Theorem
  - Divergence Theorem
  - Dyadic tensors
- Conservation of mass, momentum, and energy
- The frozen-in condition
- Helicity and cross-helicity

# Stokes' Theorem

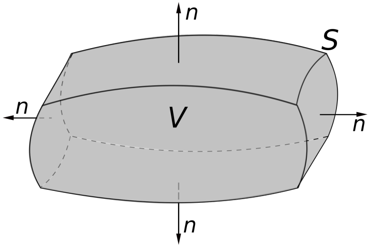


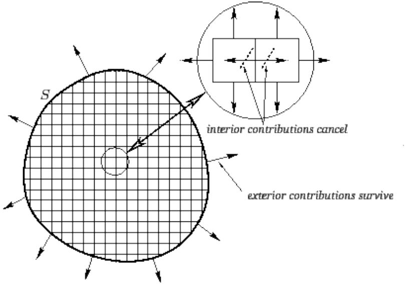

- Let $\mathcal{S}$ be an oriented piecewise-smooth surface that is bounded by a simple, close, piecewise-smooth boundary curve $\mathcal{C}$ with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $\mathcal{S}$. Then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} \tag{1}$$

where $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\,\mathrm{d}\mathcal{S}$ and $\hat{\mathbf{n}}$ is a unit normal vector to $\mathcal{S}$

# The Divergence Theorem (also called Gauss's Theorem)




- Let $\mathcal{V}$ be a simple solid region and let $\mathcal{S}$ be the boundary surface of $\mathcal{V}$, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $\mathcal{V}$. Then

$$\oint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{\mathcal{V}} \nabla \cdot \mathbf{F} \, \mathrm{d}\mathcal{V} \tag{2}$$

- The interior contributions to the divergence integral cancel, so only the exterior contributions remain

## While somewhat horrible, dyadic tensors[1] allow the equations of MHD to be written in a compact/useful form

- A dyadic tensor is a second rank tensor that can be written as the tensor product of two vectors: $A_{ij} = B_i C_j$

$$\begin{aligned}
\mathbf{A} = \mathbf{BC} &= \left(B_1 \hat{\boldsymbol{e}}_\mathbf{1} + B_2 \hat{\boldsymbol{e}}_\mathbf{2} + B_3 \hat{\boldsymbol{e}}_\mathbf{3}\right)\left(C_1 \hat{\boldsymbol{e}}_\mathbf{1} + C_2 \hat{\boldsymbol{e}}_\mathbf{2} + C_3 \hat{\boldsymbol{e}}_\mathbf{3}\right) \\
&= B_1 C_1 \hat{\boldsymbol{e}}_\mathbf{1}\hat{\boldsymbol{e}}_\mathbf{1} + B_1 C_2 \hat{\boldsymbol{e}}_\mathbf{1}\hat{\boldsymbol{e}}_\mathbf{2} + B_1 C_3 \hat{\boldsymbol{e}}_\mathbf{1}\hat{\boldsymbol{e}}_\mathbf{3} + \\
&\quad B_2 C_1 \hat{\boldsymbol{e}}_\mathbf{2}\hat{\boldsymbol{e}}_\mathbf{1} + B_2 C_2 \hat{\boldsymbol{e}}_\mathbf{2}\hat{\boldsymbol{e}}_\mathbf{2} + B_2 C_3 \hat{\boldsymbol{e}}_\mathbf{2}\hat{\boldsymbol{e}}_\mathbf{3} + \\
&\quad B_3 C_1 \hat{\boldsymbol{e}}_\mathbf{3}\hat{\boldsymbol{e}}_\mathbf{1} + B_3 C_2 \hat{\boldsymbol{e}}_\mathbf{3}\hat{\boldsymbol{e}}_\mathbf{2} + B_3 C_3 \hat{\boldsymbol{e}}_\mathbf{3}\hat{\boldsymbol{e}}_\mathbf{3}
\end{aligned} \tag{3}$$

where $\hat{\boldsymbol{e}}_\mathbf{1}\ldots$ are unit vectors and $\hat{\boldsymbol{e}}_\mathbf{1}\hat{\boldsymbol{e}}_\mathbf{3}\ldots$ are unit dyads, e.g.:

$$\hat{\boldsymbol{e}}_\mathbf{1}\hat{\boldsymbol{e}}_\mathbf{3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

[1] Good resources: *Lectures in Magnetohydrodynamics* by Dalton Schnack and `http://people.rit.edu/pnveme/EMEM851n/constitutive/tensors_rect.html`

# Dot products and double dot products with dyadic tensors

- In general, $\mathbf{BC} \neq \mathbf{CB}$
- Dot products from the left and right are different:

$$\hat{\boldsymbol{e}}_{\mathbf{1}} \cdot \mathbf{A} = B_1 C_1 \hat{\boldsymbol{e}}_{\mathbf{1}} + B_1 C_2 \hat{\boldsymbol{e}}_{\mathbf{2}} + B_1 C_3 \hat{\boldsymbol{e}}_{\mathbf{3}} \quad (5)$$

$$\mathbf{A} \cdot \hat{\boldsymbol{e}}_{\mathbf{1}} = B_1 C_1 \hat{\boldsymbol{e}}_{\mathbf{1}} + B_2 C_1 \hat{\boldsymbol{e}}_{\mathbf{2}} + B_3 C_1 \hat{\boldsymbol{e}}_{\mathbf{3}} \quad (6)$$

- Double-dot notation is ambiguous but we use:

$$\mathbf{A} : \mathbf{T} \equiv A_{ij} T_{ij}$$

where we sum over repeated indices

## The divergence of a dyadic tensor is a vector

- The divergence of a dyadic tensor is

$$\nabla \cdot \mathbf{T} = \partial_i \mathbf{T}_{ij} \tag{7}$$

  This is a vector whose $j$th component is

$$(\nabla \cdot \mathbf{T})_j = \frac{\partial T_{1j}}{\partial x_1} + \frac{\partial T_{2j}}{\partial x_2} + \frac{\partial T_{3j}}{\partial x_3} \tag{8}$$

- We can even generalize Gauss' theorem!

$$\int \nabla \cdot \mathbf{T} \, \mathrm{d}\mathcal{V} = \oint_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \mathbf{T} \tag{9}$$

# What about the dyad $\mathbf{T} \equiv \nabla \mathbf{V}$?

- The gradient operator is given by

$$\nabla \equiv \hat{\boldsymbol{e}}_{\boldsymbol{i}} \frac{\partial}{\partial x_i} \equiv \hat{\boldsymbol{e}}_{\mathbf{1}} \frac{\partial}{\partial x_1} + \hat{\boldsymbol{e}}_{\mathbf{2}} \frac{\partial}{\partial x_2} + \hat{\boldsymbol{e}}_{\mathbf{3}} \frac{\partial}{\partial x_3} \tag{10}$$

- Then $\mathbf{T} \equiv \nabla \mathbf{V}$ is given in Cartesian coordinates by

$$(\nabla \mathbf{V})_{ij} = \frac{\partial V_j}{\partial x_i} \tag{11}$$

$$\nabla \mathbf{V} = \frac{\partial V_j}{\partial x_i} \hat{\boldsymbol{e}}_{\boldsymbol{i}} \hat{\boldsymbol{e}}_{\boldsymbol{j}} \tag{12}$$

$$= \begin{pmatrix} \partial_x V_x & \partial_x V_y & \partial_x V_z \\ \partial_y V_x & \partial_y V_y & \partial_y V_z \\ \partial_z V_x & \partial_z V_y & \partial_z V_z \end{pmatrix} \tag{13}$$

where $i$ is the row index and $j$ is the column index

# Conservative form

- As mentioned previously, conservative form is usually given by

$$\frac{\partial}{\partial t}(\text{stuff}) + \nabla \cdot (\text{flux of stuff}) = 0 \tag{14}$$

  where source and sink terms go on the RHS
  - If 'stuff' is a scalar, then 'flux of stuff' is a vector
  - If 'stuff' is a vector, then 'flux of stuff' is a dyadic tensor
- If Eq. 14 is satisfied, then 'stuff' is locally conserved

# The continuity equation is already in conservative form

- The continuity equation[2] is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{15}$$

  where $\rho \mathbf{V}$ is the mass flux
- Mass is locally conserved $\Rightarrow$ the universe is safe once again!

[2] This expression is similar to the Law of Conservation of Space Wombats

# Is mass conserved globally?

- Integrate the continuity equation over a closed volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$

$$\int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V})\right] \mathrm{d}\mathcal{V} = 0 \qquad (16)$$

$$\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho \, \mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \nabla \cdot (\rho \mathbf{V}) \, \mathrm{d}\mathcal{V} = 0 \qquad (17)$$

- Now define $M$ as the mass within $\mathcal{V}$ and use Gauss' theorem

$$\underbrace{\frac{\mathrm{d}M}{\mathrm{d}t}}_{\text{change of mass in } \mathcal{V}} + \underbrace{\oint_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot (\rho \mathbf{V})}_{\text{mass flow through } \mathcal{S}} = 0 \qquad (18)$$

- Yep, we're still good. Mass is conserved globally.
- Recall that this was how we derived the continuity equation in the first place!

# The momentum equation in conservative form

- The momentum equation can be written as

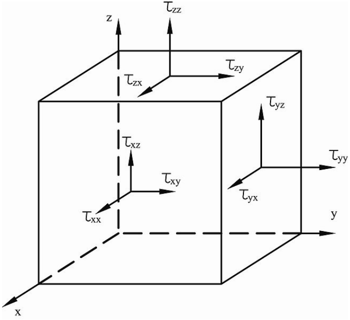
$$\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot \mathbf{T} = 0 \tag{19}$$

  where the ideal MHD stress tensor $\mathbf{T}$ is

$$\mathbf{T} = \rho \mathbf{V}\mathbf{V} + \left( p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \tag{20}$$

  and $\mathbf{I} \equiv \hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2 \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_3$ is the identity dyadic tensor
- The quantity $p + B^2/8\pi$ is the total pressure
- One can also include the viscous and gravitational stress tensors, or use a pressure tensor rather than a scalar pressure

# Physical meaning of the stress tensor




- Imagine you have an infinitesimal box
- Forces are exerted on each face by the outside volume
- The forces on each side of the box could have components in three directions
- The stress tensor includes the nine quantities needed to describe the total force exerted on this box

# The Reynolds stress tensor

- The Reynolds stress is given by

$$\mathbf{T}_V \equiv \rho \mathbf{V}\mathbf{V} \tag{21}$$

- $\mathbf{T}_V$ represents the flux of momentum. Think:

$$\mathbf{T}_V \equiv \underbrace{\rho \mathbf{V}}_{\text{Momentum density}} \underbrace{\mathbf{V}}_{\text{times a velocity}} \tag{22}$$

- $(\rho V_x) V_y$ is the rate at which the $x$ component of momentum is carried in the $y$ direction (and vice versa)

# The Maxwell stress tensor

- The Maxwell stress is

$$\mathbf{T}_B \equiv \frac{B^2}{8\pi}\mathbf{I} - \frac{\mathbf{BB}}{4\pi} \tag{23}$$

- Momentum is transported by the magnetic field
- The quantity $\frac{\mathbf{BB}}{4\pi}$ is the hoop stress which resists shearing motions

# We can't always drop gravity! (Kulsrud, p. 77)

- Gravity is often treated as a body force where $\mathbf{g}$ is constant
- More generally, $\mathbf{g}$ is found using the gradient of a scalar potential, $\phi$,

$$\mathbf{g} = -\nabla\phi \tag{24}$$

  where Poisson's equation is used to solve for the potential,

$$\nabla^2\phi = 4\pi G\rho \tag{25}$$

- The gravitational stress tensor is

$$\mathbf{T}_g = \frac{(\nabla\phi)^2\,\mathbf{I}}{8\pi G} - \frac{\nabla\phi\nabla\phi}{4\pi G} \tag{26}$$

  where $\nabla\phi\nabla\phi$ is a dyadic tensor
- The usual form of the force may be recovering using

$$\rho\mathbf{g} = -\nabla\cdot\mathbf{T}_g \tag{27}$$

# CGL theory allows for $p_{\parallel} \neq p_{\perp}$

- For a scalar pressure, the corresponding part of the stress tensor is symmetric

$$\mathbf{T}_p = p\mathbf{I} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \tag{28}$$

- Chew, Goldberger, & Low (1956) allow for different perpendicular and parallel pressures in the 'double adiabatic' approximation

$$\mathbf{T}_{CGL} = \begin{pmatrix} p_{\perp} & 0 & 0 \\ 0 & p_{\perp} & 0 \\ 0 & 0 & p_{\parallel} \end{pmatrix} \tag{29}$$

- Used more frequently in astrophysics than in laboratory or space plasma physics

# Conservation of energy

- The energy equation can be written as

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{s} = 0 \tag{30}$$

- The energy density is

$$w = \underbrace{\frac{\rho V^2}{2}}_{\text{kinetic}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}} + \underbrace{\frac{p}{\gamma - 1}}_{\text{internal}} \tag{31}$$

- The energy flux is

$$\mathbf{s} = \underbrace{\frac{\rho V^2}{2}\mathbf{V}}_{\text{kinetic}} + \underbrace{\frac{p}{\gamma - 1}\mathbf{V}}_{\text{internal}} + \underbrace{p\mathbf{V}}_{\text{work}} + \underbrace{\frac{c\mathbf{E} \times \mathbf{B}}{4\pi}}_{\text{Poynting}} \tag{32}$$

- $p\mathbf{V}$ represents the work done on the plasma from $-\nabla p$ forces

# We now have a set of three equations describing conservation of mass, momentum, and energy

- Conservation of mass, momentum, & energy are given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{33}$$

$$\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot \mathbf{T} = 0 \tag{34}$$

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{s} = 0 \tag{35}$$

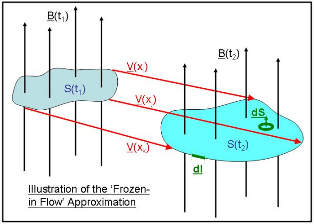
where the stress tensor, energy density, and energy flux are

$$\mathbf{T} = \rho \mathbf{V}\mathbf{V} + \left( p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \tag{36}$$

$$w = \frac{\rho V^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \tag{37}$$

$$\mathbf{s} = \left( \frac{\rho V^2}{2} + \frac{\gamma}{\gamma - 1} p \right) \mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \tag{38}$$

# The most important property of ideal MHD is the *frozen-in condition*




- Define $\mathcal{S}$ as a surface bounded by a closed curve $\mathcal{C}$ that is co-moving with the plasma
- The magnetic flux through $\mathcal{S}$ is

$$\Psi = \int_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \tag{39}$$

- There are contributions to $\frac{\mathrm{d}\Psi}{\mathrm{d}t}$ from:
  - Changes in $\mathbf{B}$ with $\mathcal{S}$ held fixed, $\frac{\mathrm{d}\Psi_1}{\mathrm{d}t}$
  - The flux swept out by $\mathcal{C}$ as it moves with the plasma, $\frac{\mathrm{d}\Psi_2}{\mathrm{d}t}$

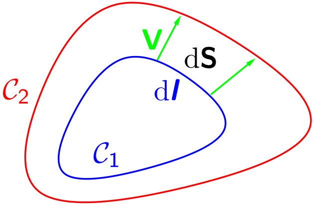
# How does the flux change due to $\mathbf{B}$ changing?

- Take the time derivative of the magnetic flux with $\mathcal{S}$ fixed:

$$\begin{aligned}
\frac{\mathrm{d}\Psi_1}{\mathrm{d}t} &= \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} \\
&= -c \int_{\mathcal{S}} (\nabla \times \mathbf{E}) \cdot \mathrm{d}\mathbf{S} \\
&= -c \oint_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{l} \qquad (40)
\end{aligned}$$

where we used Faraday's law and Stokes' theorem.

# How does the area of $\mathcal{S}$ change as it is swept with the plasma?




- The lateral area swept out by $\mathcal{C}$ as it flows with the plasma is

$$\mathrm{d}\mathbf{S} = \mathbf{V}\mathrm{d}t \times \mathrm{d}\boldsymbol{l} \tag{41}$$

  where the line element $\mathrm{d}\boldsymbol{l}$ is tangent to $\mathcal{C}$

- The flux through this area is $\mathbf{B} \cdot \mathrm{d}\mathbf{S} = \mathbf{B} \cdot \mathbf{V}\mathrm{d}t \times \mathrm{d}\boldsymbol{l}$, so that

$$\begin{aligned} \frac{\mathrm{d}\Psi_2}{\mathrm{d}t} &= \oint_{\mathcal{C}} \mathbf{B} \cdot \mathbf{V} \times \mathrm{d}\boldsymbol{l} \\ &= -\oint_{\mathcal{C}} \mathbf{V} \times \mathbf{B} \cdot \mathrm{d}\boldsymbol{l} \end{aligned} \tag{42}$$

# Putting it all together

- The total change in magnetic flux through $\mathcal{S}$ is then

$$\begin{aligned}
\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= \frac{\mathrm{d}\Psi_1}{\mathrm{d}t} + \frac{\mathrm{d}\Psi_2}{\mathrm{d}t} \\
&= -\underbrace{c \oint_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\boldsymbol{l}}_{\text{changes in } \mathbf{B}} - \underbrace{\oint_{\mathcal{C}} \mathbf{V} \times \mathbf{B} \cdot \mathrm{d}\boldsymbol{l}}_{\text{changes in } \mathcal{S}} \\
&= -c \oint_{\mathcal{C}} \left( \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right) \cdot \mathrm{d}\boldsymbol{l} \qquad (43)
\end{aligned}$$

- But $\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0$ in ideal MHD, so that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = 0. \qquad (44)$$

The magnetic flux through any co-moving fluid element remains constant in ideal MHD. ■

# In ideal MHD, the magnetic field and plasma are *frozen-in* to each other

- If a parcel of plasma moves, the magnetic field attached to the parcel moves along with it
- More rigorously: if two plasma elements are initially connected by a magnetic field line, they will remain connected by a magnetic field line at future times.
- Magnetic topology (e.g., connectivity) is preserved in ideal MHD
- The plasma cannot move across magnetic field lines (though it remains free to move along field lines)

# A useful concept in ideal MHD is a *field line velocity*

- More generally, flux will be frozen-in if there exists some velocity $\mathbf{U}$ such that

$$\mathbf{E} + \frac{\mathbf{U} \times \mathbf{B}}{c} = 0 \tag{45}$$

  The velocity $\mathbf{U}$ is a field line velocity.
- In ideal MHD, $\mathbf{U}$ is the bulk fluid velocity $\mathbf{V}$
- A field line velocity does not need to be a fluid velocity
- In Hall MHD, the field line velocity is the electron velocity

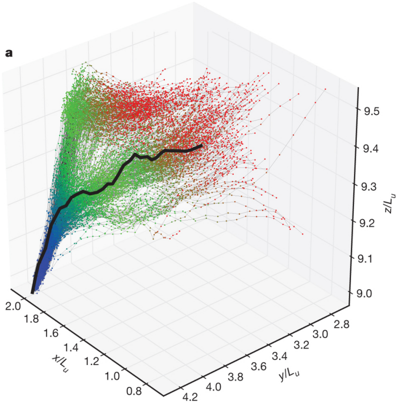
# What is meant by a field line velocity?

- In ideal MHD, a key field line velocity is

$$\mathbf{V}_{\perp} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{46}$$

  This is the component of velocity perpendicular to **B** that the plasma is traveling at which is frozen into **B**
- The concept of a field line velocity is frought with peril.
  - There is no way to distinguish one field line from another at different times
  - There are an infinite number of field line velocities
    - $\mathbf{U}_{\parallel}$ is arbitrary (but in ideal MHD we usually pick $\mathbf{V}_{\parallel}$)

# Eyink et al. (2013) argue that stochastic field line wandering in turbulence leads to Richardson diffusion that breaks down the frozen-in condition

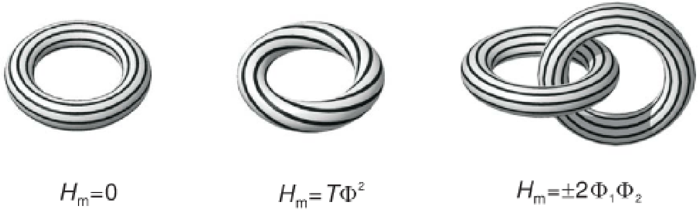



- This result warrants further investigation by independent groups

# Magnetic helicity measures the linkage of magnetic fields

- The magnetic field $\mathbf{B}$ can be written in terms of a vector potential
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{47}$$
while noting that $\mathbf{A}$ is not gauge invariant: $\mathbf{A}' = \mathbf{A} + \nabla\phi$
- The helicity of a magnetic field is given by
$$H = \int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, \mathrm{d}\mathcal{V} \tag{48}$$
- $\mathbf{A} \cdot \mathbf{B}$ should not be considered a 'helicity density' because of gauge freedom!
- Helicity is approximately conserved during magnetic reconnection and topology changes
- Helicity can be injected into a system such as the solar corona. When too much builds up, helicity ends up being expelled through coronal mass ejections.

# Helicity examples




- A single untwisted closed flux loop has $H = 0$
- A single flux rope with a magnetic flux of $\Phi$ that twists around itself $T$ times has a helicity of $H = T\Phi^2$
- Two interlinked untwisted flux loops with fluxes $\Phi_1$ and $\Phi_2$ have $H = \pm 2\Phi_1\Phi_2$ where the sign depends on the sense of the linkedness

# There are generalizations to allow for gauge-invariant definitions of helicity

- Berger & Field (1984) defined the relative magnetic helicity to be
$$H = \int_{\mathcal{V}\infty} (\mathbf{A} \cdot \mathbf{B} - \mathbf{A}_0 \cdot \mathbf{B}_0)\, \mathrm{d}\mathcal{V} \tag{49}$$
where $\mathbf{B}_0 = \nabla \times \mathbf{A}_0$ is the potential field inside $\mathcal{V}$ with the same field outside of $\mathcal{V}$ (see also Finn & Antonsen 1985)
- In toroidal laboratory experiments, it is natural to consider the volume contained within conducting wall boundaries that are coincident with closed flux surfaces (i.e., the magnetic field along the wall is parallel to the boundary)

# Helicity changes through parallel electric fields and helicity being added/removed to system

- The time evolution of magnetic helicity is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -2c \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{B} \, \mathrm{d}\mathcal{V} + 2c \int_{\mathcal{S}} \mathbf{A}_p \times \mathbf{E} \cdot \mathrm{d}\mathbf{S} \qquad (50)$$

  where we choose $\nabla \times \mathbf{A}_p = 0$ and $\mathbf{A}_p \cdot \mathrm{d}\mathbf{S} = 0$ on $\mathcal{S}$
- The first term represents helicity dissipation when $E_\parallel \neq 0$
  - But in ideal MHD, $E_\parallel = 0$!
- The second term represents helicity fluxes in and out of system
  - Flux emergence from the solar photosphere corresponds to helicity injection in the corona
  - Helicity injection in laboratory experiments [e.g., the Steady Inductive Helicity Injected Torus (HIT-SI)]

## The cross helicity $H_C$ measures the imbalance between interacting waves (important in MHD turbulence)

- The cross helicity is given by

$$H_C = \int_{\mathcal{V}} \mathbf{V} \cdot \mathbf{B} \, \mathrm{d}\mathcal{V} \tag{51}$$

- In ideal MHD, the rate of change of $H_C$ is

$$\frac{\mathrm{d}H_C}{\mathrm{d}t} = -\oint_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \left[ \left( \frac{1}{2} V^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \right) \mathbf{B} - \mathbf{V} \times (\mathbf{V} \times \mathbf{B}) \right] \tag{52}$$

  This vanishes when $\mathrm{d}\mathbf{S} \cdot \mathbf{B} = \mathrm{d}\mathbf{S} \cdot \mathbf{V} = 0$ along the boundary $\mathcal{S}$ or when the boundary conditions are periodic.
- Cross helicity is an ideal MHD invariant when this integral vanishes

# Summary

- Ideal MHD conserves mass, momentum, and energy (as it darn well should!)
- In ideal MHD, the magnetic field is *frozen-in* to the plasma flow
- It is often useful to think in terms of a magnetic field line velocity, but there are caveats!
- Helicity and cross-helicity provide two topological constraints that are conserved in ideal MHD