Ideal Magnetohydrodynamics (MHD)

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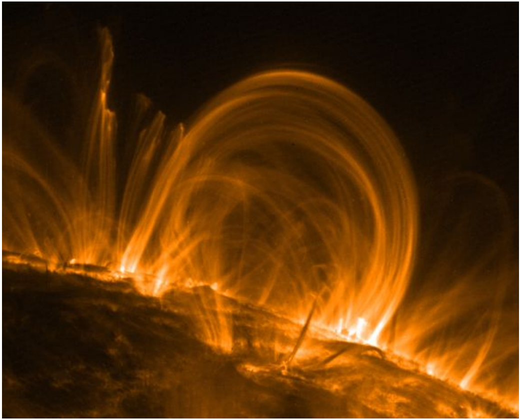
These lecture notes are largely based on *Lectures in Magnetohydrodynamics* by the late Dalton Schnack, *Ideal Magnetohydrodynamics* by Jeffrey Freidberg, *Plasma Physics for Astrophysics* by Russell Kulsrud, *Magnetic Reconnection* by Eric Priest and Terry Forbes, course notes from similar classes taught by Ellen Zweibel and Chris Hegna, and a picture of flying wombats that I found on the internet

Outline

- Introducing ourselves (since we forgot to on the first day!)
- Discuss location of course
- Overview of MHD
  - Approximation
  - Usefulness
  - Applications
- The equations of MHD and their physical meaning
  - Continuity equation
  - Momentum equation
  - Energy equation
  - Faraday's law
  - Ohm's law

What is MHD?



MHD couples Maxwell's equations with hydrodynamics to describe the macroscopic behavior of highly conducting fluids such as plasmas

Ideal MHD at a glance (cgs units)

Continuity Equation	$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0$
Momentum Equation	$\rho\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} = \frac{\mathbf{J}\times\mathbf{B}}{c} - \nabla p$
Ampere's law	$\mathbf{J} = \frac{c}{4\pi}\nabla\times\mathbf{B}$
Faraday's law	$\frac{\partial\mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E}$
Ideal Ohm's law	$\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0$
Divergence constraint	$\nabla\cdot\mathbf{B} = 0$
Adiabatic Energy Equation	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = 0$

Definitions: $\mathbf{B}$, magnetic field; $\mathbf{V}$, plasma velocity; $\mathbf{J}$, current density; $\mathbf{E}$, electric field; ρ, mass density; p, plasma pressure; γ, ratio of specific heats (usually 5/3); t, time.

MHD is a low-frequency, long-wavelength approximation

- MHD is valid on time scales longer than the inverses of the plasma frequencies and cyclotron frequencies for both ions and electrons:

$$\tau \gg \omega_{pe}^{-1}, \omega_{pi}^{-1}, \Omega_{ce}^{-1}, \Omega_{ci}^{-1} \tag{1}$$

- MHD is valid on length scales longer than the Debye length and electon/ion gyroradii:

$$L \gg \lambda_D, r_{Le}, r_{Li} \tag{2}$$

- MHD assumes quasineutrality (since $L \gg \lambda_D$)

MHD is a low-frequency, long-wavelength approximation

- MHD assumes that collisions are frequent enough for the particle distribution function to be Maxwellian with $T_i = T_e$
- Ideal MHD assumes an adiabatic equation of state
  - No additional heating, cooling, or dissipation
- MHD assumes that the plasma is fully ionized
- MHD ignores the most important advances in physics since $\sim$1860
  - Ignore relativity (assume $V^2 \ll c^2$)
  - Ignore quantum mechanics
  - Ignore displacement current in Ampere's law (assume $V^2 \ll c^2$)

When is MHD useful?

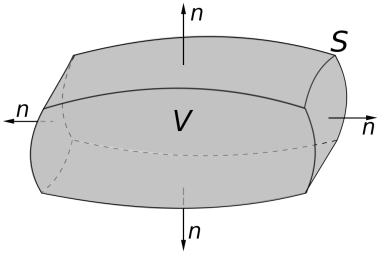
- MHD traditionally describes macroscopic force balance, equilibria, and dynamics
- MHD is a good predictor of stability
  - The most catastrophic instabilities are unstable in ideal MHD
  - Important in laboratory plasmas, solar atmosphere, etc.
- Systems that are often described using MHD include:
  - Solar wind, heliosphere, and Earth's magnetosphere[1]
  - Inertial range of plasma turbulence
  - Neutron star magnetospheres
- MHD is a reasonably good approximation in many astrophysical plasmas
  - However, extensions are often needed

[1] On large scales!

When is MHD not useful?

- MHD has limited applicability when:
  - Non-fluid or kinetic effects are important
    - Dissipation in the turbulent solar wind
    - Magnetic reconnection
    - Small-scale dynamics in Earth's magnetosphere
  - Particle distribution functions are non-Maxwellian
    - Cosmic rays
  - The plasma is weakly ionized
    - Solar photosphere/chromosphere, molecular clouds, protoplanetary disks, Earth's ionosphere, some laboratory plasmas
- MHD is mediocre at describing the dynamics of laboratory plasmas but remains a good predictor of stability

Deriving the continuity equation



- Pick a closed volume $\mathcal{V}$ bounded by a fixed surface $\mathcal{S}$ containing plasma with mass density ρ
- The total mass contained in the volume is

$$M = \int_{\mathcal{V}} \rho \, \mathrm{d}\mathcal{V} \tag{3}$$

- The time derivative of the mass in $\mathcal{V}$ is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}\mathcal{V} \tag{4}$$

The continuity equation describes conservation of mass

- The mass flowing through a surface element $\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\mathrm{d}\mathcal{S}$ is $\rho\mathbf{V} \cdot \mathrm{d}\mathbf{S}$, where the unit vector $\hat{\mathbf{n}}$ is pointing outward
- The integral of $\rho\mathbf{V} \cdot \mathrm{d}\mathbf{S}$ must equal $-\mathrm{d}M/\mathrm{d}t$:

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \mathrm{d}\mathcal{V} = - \oint_{\mathcal{S}} \rho\mathbf{V} \cdot \mathrm{d}\mathbf{S} \tag{5}$$

  This says that the change in mass inside $\mathcal{V}$ equals the mass entering or leaving the surface.
- Using Gauss' theorem we arrive at

$$\int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{V}) \right] \mathrm{d}\mathcal{V} = 0 \tag{6}$$

- This must be true for all possible volumes so the integrand must equal zero

The continuity equation in conservative form

- The continuity equation in conservative form is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{7}$$

- Conservative form is usually given by

$$\frac{\partial}{\partial t}(\text{stuff}) + \nabla \cdot (\text{flux of stuff}) = 0 \tag{8}$$

- Source and sink terms go on the RHS
  - *Example:* In a partially ionized plasma, there are continuity equations for both the ions and neutrals. Ionization acts as a source term in the ion continuity equation and a sink term in the neutral continuity equation.
- The mass flux is given by $\rho \mathbf{V}$

The second golden rule of astrophysics

The *density* of wombats

times the *velocity* of wombats

gives the *flux* of wombats.

The continuity equation

- Using vector identities we may write the continuity equation as

$$\frac{\partial \rho}{\partial t} + \underbrace{\mathbf{V} \cdot \nabla \rho}_{\text{advection}} = \underbrace{-\rho \nabla \cdot \mathbf{V}}_{\text{compression}} \tag{9}$$

- The advective derivative $\mathbf{V} \cdot \nabla \rho$ is a directional derivative that measures the change of ρ in the direction of $\mathbf{V}$
- The compression term
  - $\nabla \cdot \mathbf{V} < 0 \iff$ converging flow $\iff$ compression
  - $\nabla \cdot \mathbf{V} > 0 \iff$ diverging flow $\iff$ dilation
  - $\nabla \cdot \mathbf{V} \equiv 0 \iff$ the plasma is incompressible

The advective derivative $\mathbf{V} \cdot \nabla$ is used to describe the spatial variation of a field in the direction of the flow

- For a scalar quantity φ, the advective derivative is given by

$$\mathbf{V} \cdot \nabla\varphi = V_x \frac{\partial\varphi}{\partial x} + V_y \frac{\partial\varphi}{\partial y} + V_z \frac{\partial\varphi}{\partial z}, \tag{10}$$

which is also a scalar.

Advective derivatives in vector fields

- For a vector field $\mathbf{F}$, the advective derivative may be treated as either $(\mathbf{V}\cdot\nabla)\,\mathbf{F}$ or as the tensor derivative $\mathbf{V}\cdot(\nabla\mathbf{F})$. Both forms are equivalent, but $(\mathbf{V}\cdot\nabla)\,\mathbf{F}$ is easier to work with.

$$
\begin{aligned}
(\mathbf{V}\cdot\nabla)\,\mathbf{F} &= \left[\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}\right]\mathbf{F} \\
&= (V_x\partial_x + V_y\partial_y + V_z\partial_z)\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \\
&= \begin{pmatrix} V_x\partial_x F_x + V_y\partial_y F_x + V_z\partial_z F_x \\ V_x\partial_x F_y + V_y\partial_y F_y + V_z\partial_z F_y \\ V_x\partial_x F_z + V_y\partial_y F_z + V_z\partial_z F_z \end{pmatrix} \qquad (11)
\end{aligned}
$$

- The order of operations does not matter so we write $\mathbf{V}\cdot\nabla\mathbf{F}$
- See the *NRL Plasma Formulary* for curvilinear coordinates

The Eulerian and Lagrangian forms of the continuity equation are equivalent

- The Eulerian form follows the density at a fixed location in space:
$$\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{V} \tag{12}$$
- The Lagrangian form allows us to follow a volume element that is co-moving with the fluid:
$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \mathbf{V} = 0 \tag{13}$$
where the total derivative is given by
$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \tag{14}$$
and measures the change of a quantity as we move with the fluid. The advective derivative links the Eulerian and Lagrangian forms.

The momentum equation is derived from Newton's 2nd law

- Newton's second law of motion for a fluid element is

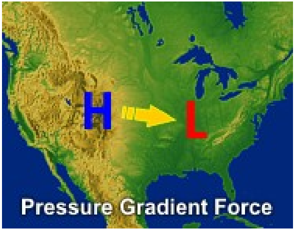
$$\rho \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F} \tag{15}$$

  where $\mathbf{F}$ is the force per unit volume
- Example forces include
  - Lorentz force: $\mathbf{F}_L = \frac{\mathbf{J}\times\mathbf{B}}{c}$
  - Pressure gradient force: $\mathbf{F}_p = -\nabla p$
  - Gravity: $\mathbf{F}_g = -\rho\mathbf{g}$ or $\mathbf{F}_g = -\nabla\phi$ for gravitational potental ϕ
  - Viscosity: $\mathbf{F}_V = \nabla \cdot \mathbf{\Pi}$, where $\mathbf{\Pi}$ is the viscous stress tensor
- The ideal MHD momentum equation in Eulerian form is

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p \tag{16}$$

  where we neglect gravity and ignore viscous forces

The pressure gradient force $-\nabla p$ pushes plasma from regions of high plasma pressure to low plasma pressure



- The pressure gradient force is orthogonal to isobars
- This is the Restoring force for sound waves
- Resulting motions are not necessarily in the direction of the pressure gradient force (e.g., when other forces are acting on the fluid)

Where does the Lorentz force come from?

- The Lorentz force acting on a single particle is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c}\right) \tag{17}$$

- The current density is given by

$$\mathbf{J} = \sum_{\alpha} n_{\alpha} q_{\alpha} \mathbf{V}_{\alpha} \tag{18}$$

where α includes all species of ions and electrons. For a quasineutral plasma with electrons and singly charged ions, this becomes

$$\mathbf{J} = en\left(\mathbf{V}_i - \mathbf{V}_e\right) \tag{19}$$

where $n = n_e = n_i$, $\mathbf{V}_i$ is the ion velocity, and $\mathbf{V}_e$ is the electron velocity.

The Lorentz force includes a magnetic tension force and a magnetic pressure force

- Use Ampere's law and vector identities to decompose the Lorentz force term into two components

$$
\begin{aligned}
\underbrace{\frac{\mathbf{J}\times\mathbf{B}}{c}}_{\text{Lorentz force}} &= \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} \\
&= \underbrace{\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}}_{\sim\text{magnetic tension}} - \underbrace{\nabla\left(\frac{B^2}{8\pi}\right)}_{\sim\text{magnetic pressure}} \qquad (20)
\end{aligned}
$$

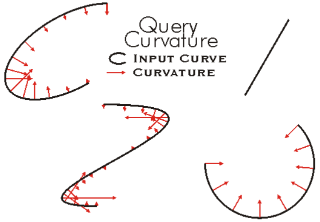
- While the Lorentz force must be orthogonal to be **B**, both of these terms may have components along **B**. The parallel component of the above tension term cancels out the parallel part of the magnetic pressure term (Kulsrud §4.2).

The curvature vector $\boldsymbol{\kappa}$ gives the rate at which the tangent vector turns

- Define $\hat{\mathbf{b}}$ as a unit vector in the direction of $\mathbf{B}$: $\hat{\mathbf{b}} \equiv \mathbf{B}/|\mathbf{B}|$
- The curvature vector $\boldsymbol{\kappa}$ points toward the center of curvature and is given by

$$\boldsymbol{\kappa} \equiv \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}} = -\frac{\mathbf{R}}{R^2} \tag{21}$$

where $\mathbf{R}$ is a vector from the center of curvature to the point we are considering. Note that $|\boldsymbol{\kappa}| = R^{-1}$ and $\boldsymbol{\kappa} \cdot \hat{\mathbf{b}} = 0$.



The Lorentz force can be decomposed into two terms with forces orthogonal to $\mathbf{B}$ using field line curvature

- Next use the product rule to obtain

$$\mathbf{B}\cdot\nabla\mathbf{B} = B\hat{\mathbf{b}}\cdot\nabla(B\hat{\mathbf{b}}) = \frac{\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\nabla)B^2}{2} + B^2\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} \tag{22}$$

- We can then write the Lorentz force as

$$\frac{\mathbf{J}\times\mathbf{B}}{c} = \underbrace{\boldsymbol{\kappa}\frac{B^2}{4\pi}}_{\text{magnetic tension}} - \underbrace{\nabla_\perp\left(\frac{B^2}{8\pi}\right)}_{\text{magnetic pressure}} \tag{23}$$

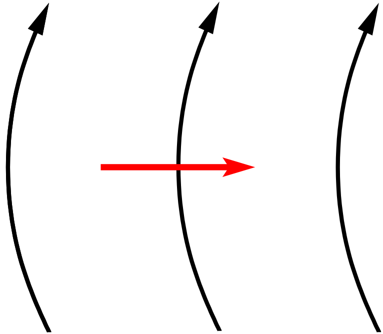
where all terms are orthogonal to $\mathbf{B}$.[2]

- The operator $\nabla_\perp$ keeps only the derivatives orthogonal to $\mathbf{B}$:

$$\nabla_\perp \equiv \nabla - \hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\nabla) \tag{24}$$

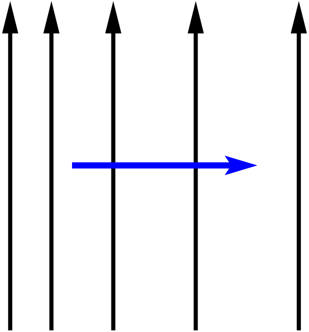
[2]Note: the terms in this formulation for magnetic tension and pressure differ from the corresponding terms in Eq. 20.

The magnetic tension force wants to straighten magnetic field lines



- The magnetic tension force is directed radially inward with respect to magnetic field line curvature

Regions of high magnetic pressure exert a force towards regions of low magnetic pressure



- The magnetic pressure is given by $p_B \equiv \frac{B^2}{8\pi}$

The ratio of the plasma pressure to the magnetic pressure is an important dimensionless number

- Define plasma β as

$$\beta \equiv \frac{\text{plasma pressure}}{\text{magnetic pressure}} \equiv \frac{p}{B^2/8\pi}$$

- If $\beta \ll 1$ then the magnetic field dominates
  - Solar corona
  - Poynting flux driven jets
  - Tokamaks ($\beta \lesssim 0.1$)
- If $\beta \gg 1$ then plasma pressure forces dominate
  - Stellar interiors
- If $\beta \sim 1$ then pressure/magnetic forces are both important
  - Solar chromosphere
  - Parts of the solar wind and interstellar medium
  - Some laboratory plasma experiments

The adiabatic energy equation provides the closure for ideal MHD

- The Lagrangian form of the adiabatic energy equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{25}$$

- The Eulerian form of the adiabatic energy equation is

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right) p = -\gamma\rho\nabla\cdot\mathbf{V} \tag{26}$$

  where the term on the RHS represents heating/cooling due to adiabatic compression/expansion.

  - The entropy of any fluid element is constant
  - Ignores thermal conduction, non-adiabatic heating/cooling

- This is generally a mediocre approximation, but is useful for some situations (e.g., MHD waves)

Faraday's law tells us how the magnetic field varies with time

Faraday's law is unchanged from Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \tag{27}$$

But how do we get the electric field?

We get the electric field from Ohm's law

- The electric field $\mathbf{E}'$ seen by a conductor moving with velocity $\mathbf{V}$ is given by

$$\mathbf{E}' = \frac{\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{28}$$

- This is Lorentz invariant, but the fluid equations are only Galilean invariant! Let's expand the denominator.

$$\begin{aligned} \mathbf{E}' &= \left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)\left(1 - \frac{1}{2}\frac{V^2}{c^2} + \dots\right) \\ &= \mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} + \mathcal{O}\left(\frac{V^2}{c^2}\right) \end{aligned} \tag{29}$$

- By setting $\mathbf{E}' = 0$, we arrive at the ideal Ohm's law

$$\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0 \tag{30}$$

which ignores $\mathcal{O}\left(\frac{V^2}{c^2}\right)$ terms and is Galilean invariant.

Ohm's law can be combined with Faraday's law for the induction equation

- Using $\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0$ and $\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$, we arrive at

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{V} \times \mathbf{B})}_{\text{advection}} \tag{31}$$

- The ideal Ohm's law neglects contributions to $\mathbf{E}$ from resistivity, the Hall effect, electron inertia, and (in partially ionized plasmas) ambipolar diffusion
- As we will soon see, the ideal Ohm's law leads to the magnetic field and plasma being frozen into each other so that magnetic topology is preserved
- Ideal MHD plasmas are perfectly conducting

The low-frequency Ampere's law

- Ampere's law without displacement current is

$$\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B} \tag{32}$$

- There is no time-dependence, so we can replace **J** in other equations using this expression
- This formulation implies that

$$\nabla \cdot \mathbf{J} = 0 \tag{33}$$

which is a necessary condition for quasineutrality
- MHD treats the plasma as a single fluid, but recall that **J** also represents the relative drift between ions and electrons

$$\mathbf{J} \equiv \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha \tag{34}$$

And of course, the most boringest of Maxwell's equations must remain satisfied

- The divergence constraint , also known as Gauss' law for magnetism. Huzzah! ...

$$\nabla \cdot \mathbf{B} = 0 \tag{35}$$

  - Magnetic monopoles do not exist
  - The magnetic charge density equals zero
  - **B** is a *solenoidal* (divergence-free) field

- We might as well put it in integral form while we're here...

$$\begin{aligned} \int_{\mathcal{V}} (\nabla \cdot \mathbf{B})\, \mathrm{d}\mathcal{V} &= 0 \\ \oint_{\mathcal{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S} &= 0 \end{aligned} \tag{36}$$

The magnetic field going into a closed volume equals the magnetic field going out of it.

If the magnetic field is initially divergence free, then it will remain divergence free because of Faraday's law

- Take the divergence of Faraday's law:

$$
\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= -c\nabla \times \mathbf{E} \\
\nabla \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) &= \nabla \cdot (-c\nabla \times \mathbf{E}) \\
\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) &= 0
\end{aligned}
$$

  since the divergence of a curl is identically zero. Well, I guess that's kind of cool.

Writing $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$ automatically satisfies the divergence constraint

- The magnetic field can be written as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{37}$$

Take the divergence:

$$\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} = 0 \tag{38}$$

- The vector potential formulation allows gauge freedom since $\nabla \times \nabla\phi = 0$ for a scalar function ϕ. Let $\mathbf{A}' = \mathbf{A} + \nabla\phi$:

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A}' \\ &= \nabla \times \mathbf{A} + \nabla \times \nabla\phi \\ &= \nabla \times \mathbf{A} \end{aligned}$$

Summary of Ideal MHD

- MHD couples Maxwell's equations with hydrodynamics to describe macroscopic behavior in highly conducting plasmas
- MHD uses the *low-frequency, long wavelength* approximation
- Each term in the ideal MHD equations has an important physical meaning
- Extensions to MHD are often needed to describe plasma dynamics
- Next up:
  - Conservation laws
  - Virial theorem
  - Extensions to MHD
  - Waves, shocks, & instabilities
  - More space wombats

Useful References on MHD

- *The Physics of Plasmas* by Boyd & Sanderson (Ch. 3)
- *Magnetohydrodynamics of the Sun* by Priest (Ch. 2)
- *Lectures in Magnetohydrodynamics* by Schnack
  - Lecture 2 reviews the math of plasma physics
  - Lectures 3–9 discuss the equations of MHD
- *Plasma Physics for Astrophysics* by Kulsrud (Ch. 3)