MHD Instabilities: Part I

Nick Murphy

Harvard-Smithsonian Center for Astrophysics

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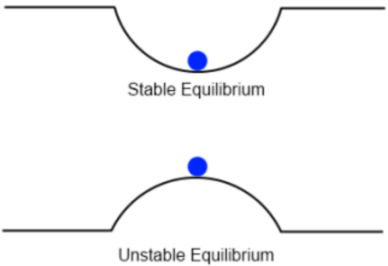
These slides are based off of Boyd & Sanderson's *The Physics of Plasmas*, Schnack's *Lectures in Magnetohydrodynamics*, Kulsrud's *Plasma Physics for Astrophysics*, Freidberg's *Ideal Magnetohydrodynamics*, and slides from Andrei Simakov.

Outline

- Overview of instabilities
- Initial value formulation
- Normal mode formulation
- Variational formulation
- Energy principle

Basic idea behind physical instabilities



- If you perturb this stable equilibrium, the ball will return to its original location, or oscillate around it if damping is weak
- If you perturb this unstable equilibrium, the ball will roll away

Let's take this analogy further

- If $W(x)$ is the potential energy, then the particle feeds a force $F_x = -\partial W/\partial x$
- Equilibria occur when $\partial W/\partial x = 0$
- To lowest order in the Taylor expansion about an equilibrium point, the change in potential energy is

$$\delta W = \frac{1}{2}\left(\frac{\partial^2 W}{\partial x^2}\right)(\Delta x)^2 \tag{1}$$

- The equilibria are stable when $\delta W > 0$
- The equilibria are unstable when $\delta W < 0$

Why study plasma instabilities?

- Plasmas are host to numerous instabilities
- An equilibrium will not exist for long if it is unstable
- Plasma instabilities are a source of turbulence which in turn can drive transport
  - Magnetorotational instability in accretion disks
  - Various short wavelength instabilities in fusion devices
- Instabilities often play an important role in energy conversion
- Plasma instabilities are responsible for a number of energetic phenomena (stellar flares, coronal mass ejections, etc.)
- In magnetically confined fusion plasmas, instabilities are a key barrier to good confinement

One could easily fill a full course on plasma instabilities

List of plasma instabilities [edit]

- Bennett pinch instability (also called the z-pinch instability)
- Beam acoustic instability
- Bump-in-tail instability
- Buneman instability,[2]
- Cherenkov instability,[3]
- Chute instability
- Coalescence instability,[4]
- Collapse instability
- Counter-streaming instability
- Cyclotron instabilities, including:
  - Alfven cyclotron instability
  - Electron cyclotron instability
  - Electrostatic ion cyclotron Instability
  - Ion cyclotron instability
  - Magnetoacoustic cyclotron instability
  - Proton cyclotron instability
  - Nonresonant Beam-Type cyclotron instability
  - Relativistic ion cyclotron instability
  - Whistler cyclotron instability
- Diocotron instability,[5] (similar to the Kelvin-Helmholtz fluid instability).
- Disruptive instability (in tokamaks)
- Double emission instability
- Drift wave instability
- Edge-localized modes [6]
- Electrothermal instability
- Farley-Buneman instability,[7]
- Fan instability
- Filamentation instability
- Firehose instability (also called Hose instability)
- Flute instability
- Free electron maser instability
- Gyrotron instability
- Helical instability (helix instability)
- Helical kink instability
- Hose instability (also called Firehose instability)
- Interchange instability
- Ion beam instability
- Kink instability
- Lower hybrid (drift) instability (in the Critical ionization velocity mechanis
- Magnetic drift instability
- Magnetorotational instability (in accretion disks)
- Magnetothermal instability (Laser-plasmas) [8]
- Modulation instability
- Non-Abelian instability (see also Chromo-Weibel instability)
- Chromo–Weibel instability
- Non-linear coalescence instability
- Oscillating two stream instability, see two stream instability
- Pair instability
- Parker instability (magnetic buoyancy instability)
- Peratt instability (stacked toroids)
- Pinch instability
- Sausage instability
- Slow Drift Instability
- Tearing mode instability
- Two-stream instability
- Weak beam instability
- Weibel instability
- z-pinch instability, also called Bennett pinch instability

Why MHD?

- Ideal MHD is frequently used to describe macroscopic instabilities
- MHD is often a mediocre approximation, but it does a surprisingly good job at describing instabilities
- MHD captures most of the essential physics of force balance
  - Magnetic tension, magnetic pressure, plasma pressure
- MHD instabilities capture the most explosive behavior
- Instability theory is closely related to wave theory which both use ideal MHD as a starting point
- There exist some laboratory configurations that are stable despite being ideally unstable
- In astrophysics, heat conduction and radiative cooling can also be destabilizing

Ideal, resistive, and kinetic instabilities

- Ideal instabilities are usually the strongest and most unstable
  - Current-driven vs. pressure-driven
- Resistive instabilities are stable unless $\eta \neq 0$
  - Growth rate is usually slower than ideal instabilities
  - Often associated with magnetic reconnection
- Kinetic instabilities are often microinstabilities that occur when the distribution functions are far from Maxwellian
  - Two-stream, Weibel, Buneman, etc.
  - Important for near-Earth space plasmas, laboratory plasmas, cosmic ray interactions with ambient plasma, and dissipation of turbulence

General strategy for studying plasma stability

- Start from an initial equilibrium, e.g.,

$$\frac{\mathbf{J}_0 \times \mathbf{B}_0}{c} = \nabla p_0 \tag{2}$$

- Linearize the equations of MHD and discard higher order terms
- Slightly perturb that equilibrium
- If there exists a growing perturbation, the system is *unstable*
- If no growing perturbation exists, the system is *stable*
- Use a combination of numerical simulations, experiments, and observations to study nonlinear dynamics

Linearizing the equations of ideal MHD

- Following the procedure for waves, we represent the relevant fields as the sum of equilibrium ('0') and perturbed ('1') components

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t) \quad (3)$$

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{V}_1(\mathbf{r}) \quad (4)$$

$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + p_1(\mathbf{r}, t) \quad (5)$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t) \quad (6)$$

  where $\mathbf{V}_0 = 0$ for a static equilibrium
- Assume that the perturbed fields are much weaker than the equilibrium fields
- Use the convention that the perturbed fields vanish at $t = 0$

Linearizing the equations of ideal MHD

- To zeroeth order, a static equilibrium is given by

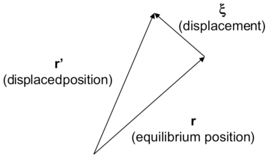
$$\nabla p_0 = \frac{(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0}{4\pi} \tag{7}$$

- Ignoring products of the perturbations gives

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\mathbf{V}_1 \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \mathbf{V}_1 && (8)\\
\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} &= -\nabla p_1 + \frac{(\nabla \times \mathbf{B}_0) \times \mathbf{B}_1}{4\pi} + \frac{(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0}{4\pi} && (9)\\
\frac{\partial \mathbf{B}_1}{\partial t} &= \frac{\nabla \times (\mathbf{V}_1 \times \mathbf{B}_0)}{c} && (10)\\
\frac{\partial p_1}{\partial t} &= -\mathbf{V}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{V}_1 && (11)
\end{aligned}$$

Note that $\mathbf{V}_1$ is the only time-dependent variable on the RHS of Eqs. 8, 10, and 11.

The displacement vector, $\boldsymbol{\xi}$, describes how much the plasma is displaced from the equilibrium state[1]



- If $\boldsymbol{\xi}(\mathbf{r}, t = 0) = 0$, then the displacement is

$$\boldsymbol{\xi}(\mathbf{r}, t) \equiv \int_0^t \mathbf{V}_1(\mathbf{r}, t')\mathrm{d}t' \tag{12}$$

- Its time derivative is just the perturbed velocity,

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{V}_1(\mathbf{r}, t) \tag{13}$$

[1] A side benefit of using slides is that I do not have to try writing $\boldsymbol{\xi}$ on the chalkboard.

Integrate the continuity equation with respect to time

- Put the linearized continuity equation in terms of $\boldsymbol{\xi}$

$$\frac{\partial \rho_1}{\partial t} = -\mathbf{V}_1 \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \mathbf{V}_1 \tag{14}$$

$$= -\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \tag{15}$$

- Next we can integrate this

$$\int_0^t \frac{\partial \rho_1}{\partial t'} \mathrm{d}t' = \int_0^t \left[-\frac{\partial \boldsymbol{\xi}}{\partial t'} \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t'} \right] \mathrm{d}t' \tag{16}$$

which leads to a solution for ρ_1 in terms of just $\boldsymbol{\xi}$

$$\rho_1(\mathbf{r}, t) = -\boldsymbol{\xi}(\mathbf{r}, t) \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \boldsymbol{\xi}(\mathbf{r}, t) \tag{17}$$

We can similarly put the linearized induction and energy equations in terms of $\boldsymbol{\xi}$

- Integrating the linearized equations with respect to time yields solutions for the perturbed density, magnetic field, and plasma pressure:

$$\begin{aligned}
\rho_1(\mathbf{r},t) &= -\boldsymbol{\xi}(\mathbf{r},t)\cdot\nabla\rho_0 - \rho_0\nabla\cdot\boldsymbol{\xi}(\mathbf{r},t) && (18)\\
\mathbf{B}_1(\mathbf{r},t) &= \nabla\times\left[\frac{\boldsymbol{\xi}(\mathbf{r},t)\times\mathbf{B}_0(\mathbf{r})}{c}\right] && (19)\\
p_1(\mathbf{r},t) &= -\boldsymbol{\xi}(\mathbf{r},t)\cdot\nabla p_0(\mathbf{r}) - \gamma p_0(\mathbf{r})\nabla\cdot\boldsymbol{\xi}(\mathbf{r},t) && (20)
\end{aligned}$$

  - The perturbed density ρ_1 doesn't appear in the other equations, which form a closed set
- However, we still have the momentum equation to worry about!

The linearized momentum equation in terms of $\boldsymbol{\xi}$ and $\mathbf{F}[\boldsymbol{\xi}]$

- Using the solutions for ρ_1, $\mathbf{B}_1$, and p_1 we arrive at

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}[\boldsymbol{\xi}(\mathbf{r}, t)] \tag{21}$$

  which looks awfully similar to Newton's second law
- The ideal MHD force operator is

$$\begin{aligned} \mathbf{F}(\boldsymbol{\xi}) &= \nabla(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}) \\ &\quad + \frac{1}{4\pi}(\nabla \times \mathbf{B}_0) \times [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0] \\ &\quad + \frac{1}{4\pi}\{[\nabla \times \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)] \times \mathbf{B}_0\} \end{aligned} \tag{22}$$

- The force operator is a function of $\boldsymbol{\xi}$ and the equilibrium, but not $\frac{\partial \boldsymbol{\xi}}{\partial t}$

Building up intuition for the force operator

- If $\boldsymbol{\xi} \cdot \mathbf{F} < 0$:
  - The displacement and force are in opposite directions
  - The force opposes displacements
  - The system will typically oscillate around the equilibrium
  - This corresponds to stability (for this displacement, anyway)
- If $\boldsymbol{\xi} \cdot \mathbf{F} > 0$:
  - The displacement and force are in the same direction
  - The force encourages displacements
  - We would expect the perturbation to grow
  - This corresponds to instability
- If $\boldsymbol{\xi} \cdot \mathbf{F} = 0$, then this perturbation is neutrally stable
- MHD does not allow *overstable* solutions where the restoring force would be strong enough to overcorrect for and amplify the oscillations
  - These would require an energy sink or source

Initial value formulation

- MHD stability can be investigated as an initial value problem by finding solutions to

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}[\boldsymbol{\xi}(\mathbf{r}, t)] \tag{23}$$

  with appropriate initial and boundary conditions
- This method yields a significant amount of information
  - Time evolution of system
  - Solution yields fastest growing mode
- This formulation is useful for complicated or non-analytic equilibria when the linearized or full MHD equations can be solved numerically
- However, this strategy usually requires extra work
- There are easier ways to get information about stability

Finding the normal mode solution

- Separate the space and time dependences of the displacement:

$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(\mathbf{r}) T(t) \tag{24}$$

- The linearized momentum equation becomes

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -\omega^2 T \tag{25}$$

$$-\omega^2 \rho_0 \boldsymbol{\xi}(\mathbf{r}) = \mathbf{F}[\boldsymbol{\xi}(\mathbf{r})] \tag{26}$$

so that $T(t) = e^{i\omega t}$ and the solution is of the form

$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(\mathbf{r}) e^{i\omega t} \tag{27}$$

- Eq. 26 is an eigenvalue problem since **F** is linear
- The BCs determine the permitted values of ω^2
  - These can be a *discrete* or *continuous* set

Finding the normal mode solution

- For a discrete set of ω^2, the normal mode solution is

$$\boldsymbol{\xi}(\mathbf{r}, t) = \sum_n \boldsymbol{\xi}_n(\mathbf{r}) e^{i\omega t} \tag{28}$$

  where $\boldsymbol{\xi}_n$ is the *normal mode* corresponding to its *normal frequency* ω_n
- Because $\mathbf{F}$ is self-adjoint, ω_n^2 must be real
- If $\omega_n^2 > 0 \; \forall n$, then the equilibrium is stable
- If $\omega_n^2 < 0$ for any n, then the equilibrium is unstable
- Stability boundaries occur when $\omega = 0$
- Now all we have to do is solve for a possibly infinite number of solutions!

The MHD force operator is *self-adjoint*

- An adjoint is a generalized complex conjugate for a functional
- The operator $\mathbf{F}(\boldsymbol{\xi})$ is *self-adjoint*. For any allowable displacement vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$

$$\int \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) \mathrm{d}\mathbf{r} = \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta}) \mathrm{d}\mathbf{r} \tag{29}$$

  For a proof, see Freidberg (1987)
- Self-adjointness is closely related to conservation of energy
  - If there is dissipation, $\mathbf{F}$ will not be self-adjoint

Showing that ω^2 is real[2]

- Start with $-\omega^2\rho\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$, dot it with $\boldsymbol{\xi}^*$, and integrate over volume

$$\omega^2 \int \rho|\boldsymbol{\xi}|^2 \mathrm{d}\mathbf{r} = -\int \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi}) \mathrm{d}\mathbf{r} \quad (30)$$

- Do the same for the complex conjugate equation $-(\omega^*)^2\rho\boldsymbol{\xi}^* = \mathbf{F}(\boldsymbol{\xi}^*)$ dotted with $\boldsymbol{\xi}$

$$(\omega^*)^2 \int \rho|\boldsymbol{\xi}|^2 \mathrm{d}\mathbf{r} = -\int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}^*) \quad (31)$$

- Subtract the two equations and use the self-adjointedness of $\mathbf{F}$

$$\left[\omega^2 - (\omega^*)^2\right] \int \rho|\boldsymbol{\xi}|^2 = 0 \quad (32)$$

This means that $\omega^2 = (\omega^*)^2$, so that ω^2 and therefore $\boldsymbol{\xi}$ are purely real!

[2] Following Simakov for these and surrounding slides

Showing that the normal modes are orthogonal

- Consider two discrete modes $(\boldsymbol{\xi}_m, \omega_m^2)$ and $(\boldsymbol{\xi}_n, \omega_n^2)$

$$-\omega_m^2 \rho \boldsymbol{\xi}_m = \mathbf{F}(\boldsymbol{\xi}_m) \tag{33}$$

$$-\omega_n^2 \rho \boldsymbol{\xi}_n = \mathbf{F}(\boldsymbol{\xi}_n) \tag{34}$$

- Dot the $\boldsymbol{\xi}_m$ equation with $\boldsymbol{\xi}_n$ and vice versa, integrate over volume, subtract, and use self-adjointedness of $\mathbf{F}$ to get

$$\left(\omega_m^2 - \omega_n^2\right) \int \rho \boldsymbol{\xi}_m \cdot \boldsymbol{\xi}_n \mathrm{d}\mathbf{r} = 0 \tag{35}$$

If $\omega_m^2 \neq \omega_n^2$ (e.g., the modes are discrete) then

$$\int \rho \boldsymbol{\xi}_m \cdot \boldsymbol{\xi}_n \mathrm{d}\mathbf{r} = 0 \tag{36}$$

The modes are orthogonal with weight function ρ!

Normal mode formulation

- This method requires less effort than the initial value formulation
- This method is more amenable to analysis
- This method cannot be used to describe nonlinear evolution (after the linearization approximation breaks down)
- Normal mode analysis requires that the eigenvalues are discrete and distinguishable
- However, there is a more elegant way to determine whether or not a system is stable
  - The energy principle!

The variational principle is the basis for the energy principle

- The kinetic energy is $\frac{1}{2}\rho_0\dot{\xi}^2$ integrated over the volume

$$K(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}) = \frac{1}{2}\int \rho_0 \dot{\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}}\,\mathrm{d}\mathbf{r} = \frac{-\omega^2}{2}\int \rho_0 \boldsymbol{\xi}\cdot\boldsymbol{\xi}\,\mathrm{d}\mathbf{r} = \frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\,\mathrm{d}\mathbf{r} \quad (37)$$

- From conservation of energy the change in potential energy is

$$\delta W(\boldsymbol{\xi},\boldsymbol{\xi}) = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\,\mathrm{d}\mathbf{r} \quad (38)$$

The variational principle

- From these equations, we arrive at

$$\omega^2 = \frac{\delta W(\boldsymbol{\xi}, \boldsymbol{\xi})}{K(\boldsymbol{\xi}, \boldsymbol{\xi})} \tag{39}$$

- Any $\boldsymbol{\xi}$ for which ω^2 is an extremum is an eigenfunction of $-\omega^2 \rho_0 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$ with eigenvalue ω^2
  - See problem 4.7 of Boyd & Sanderson
- If $\delta W < 0$, then there is an instability!

Strategy for the variational principle

- Choose a trial function

$$\boldsymbol{\xi} = \sum_n a_n \phi_n \tag{40}$$

where ϕ_n are a suitable choice of basis functions subject to the normalization condition

$$K(\boldsymbol{\xi}, \boldsymbol{\xi}) = \text{const.} \tag{41}$$

- Minimize δW with respect to the coefficients a_n
- A lower bound for the growth rate γ is

$$\gamma \geq \sqrt{-\frac{\delta W}{K}} \tag{42}$$

The energy principle

- Generally we care more about whether or not a configuration is stable than finding the linear growth rate
- The linear growth stage quickly becomes overwhelmed by nonlinear effects
- The energy principle allows us to determine stability but at the cost of losing information about the growth rate
  - We lose the restriction regarding the normalization condition

Deriving the energy principle

- For a full derivation, see Freidberg (1987). Start from

$$\delta W(\boldsymbol{\xi}, \boldsymbol{\xi}) = -\frac{1}{2} \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}) \mathrm{d}\mathbf{r} \tag{43}$$

  (Think: work equals force times distance)

- Express δW as the sum of changes in the potential energy of the plasma, δW_P, the surface, δW_S, and the vacuum, δW_V sp that

$$\delta W = \delta W_P + \delta W_S + \delta W_V \tag{44}$$

- The vacuum magnetic energy corresponds to the potential field solution for a given set of boundary conditions

Deriving the energy principle

- The plasma, surface, and vacuum contributions are

$$\delta W_P = \frac{1}{2}\int \left[\frac{B_1^2}{4\pi} - \boldsymbol{\xi}\cdot\left(\frac{\mathbf{J}_0 \times \mathbf{B}_1}{c}\right) - p_1 \left(\nabla\cdot\boldsymbol{\xi}\right)\right] \mathrm{d}\mathbf{r} \quad (45)$$

$$\delta W_S = \oint_{\mathcal{S}} (\boldsymbol{\xi}\cdot\hat{\mathbf{n}})^2 \left[\nabla\left(p_0 + \frac{B_0^2}{8\pi}\right)\right]_1^2 \cdot \mathrm{d}\mathbf{S} \quad (46)$$

$$\delta W_V = \int \frac{B_{1,vac}^2}{8\pi} \mathrm{d}\mathbf{r} \quad (47)$$

- In the case of a deformed boundary, all terms may exist. This corresponds to an *external* (free-boundary) mode.
- In the case of a fixed boundary, $\delta W_S = \delta W_V = 0$. This corresponds to an *internal* (fixed-boundary) mode.

The intuitive form of energy principle

- After manipulation the energy principle can be written as

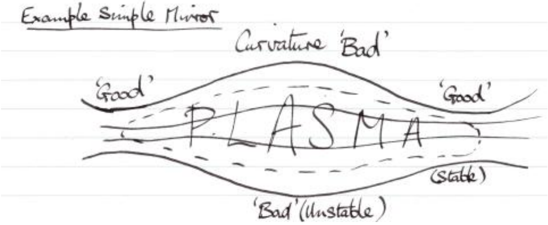
$$\delta W_P = \frac{1}{2}\int \mathrm{d}\mathbf{r}[\frac{|\mathbf{B}_{1\perp}|^2}{4\pi} + \frac{B^2}{4\pi}\left|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right|^2 + \gamma p\left|\nabla\cdot\boldsymbol{\xi}\right|^2 - 2\left(\boldsymbol{\xi}_\perp\cdot\nabla p\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right) - J_{\|}\left(\boldsymbol{\xi}_\perp^*\times\mathbf{b}\right)\cdot\mathbf{B}_{1\perp}] \qquad (48)$$

- The first three terms are always stabilizing (in order):
  - The energy required to bend magnetic field lines (shear Alfvén wave)
  - Energy necessary to compress **B** (compr. Alfvén wave)
  - The energy required to compress the plasma (sound wave)
- The remaining two terms can be stabilizing or destabilizing:
  - Pressure-driven (interchange) instabilities (associated with $\mathbf{J}_\perp$)
  - Current-driven (kink) instabilities (associated with $\mathbf{J}_|$)

Strategy for using the energy principle

- The energy principle can be used with the same strategy as the variational principle
- If there exists a trial solution $\boldsymbol{\xi}$ for which $\delta W < 0$, then the configuration is unstable
- Information on the growth rate and linear structure of the instability is lost
- However, the energy principle makes it easier to determine what is driving a particular instability

Good curvature vs. bad curvature

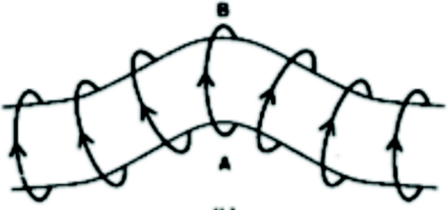


- Pressure-driven (interchange) instabilities occur when

$$\boldsymbol{\kappa} \cdot \nabla p > 0 \tag{49}$$

  where $\boldsymbol{\kappa} \equiv \hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}$ is the curvature vector. This is a necessary but not sufficient criterion for pressure-driven instabilities.
- Instability occurs when it is energetically favorable for the magnetic field and plasma to switch places
- Usually short wavelengths are most unstable

The kink instability



- When the perturbed field comes together, magnetic pressure is increased
- When perturbed field lines come apart, magnetic pressure is decreased
- Magnetic pressure continues to drive the perturbation
- The kink instability could be stabilized by magnetic field along the axis of the flux rope
  - Tension becomes a restoring force
- Usually long wavelengths are most unstable

Overstable modes

- I said that there are no overstable modes in MHD. Was I lying?
- Sort of! But we need to go beyond MHD and include, e.g.,
  - Radiative cooling
  - Anisotropic thermal conduction
- Examples include (e.g., Balbus & Reynolds 2010)
  - Magnetothermal instability
  - Heat flux buoyancy instability

  These are important in the intracluster medium of galaxy clusters.

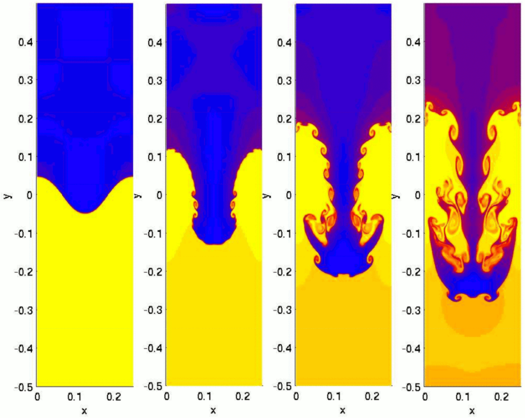
Poloidal and toroidal mode numbers

- Instabilities in toroidal laboratory plasma devices are typically described using:
  - The poloidal mode number, m
  - The toroidal mode number, n
- Instabilities in a torus are naturally periodic, and (m, n) describe how many times the instability wraps around in each of the poloidal and toroidal directions
- Usually not important in astrophysics (except perhaps in global instabilities in accretion disks) but worth mentioning!

Linear growth vs. nonlinear growth and saturation of instabilities

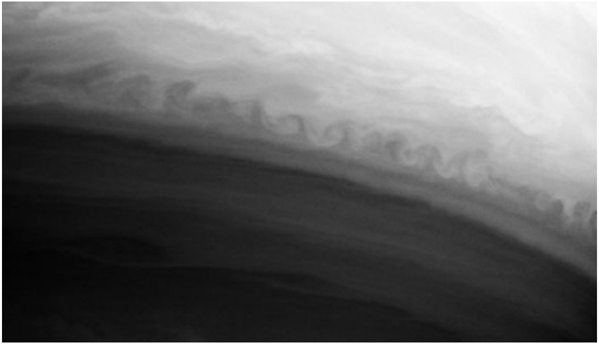
- In the initial linear stage, plasma instabilities grow exponentially
- This exponential growth continues until the linearization approximation breaks down
- Nonlinear growth is usually slower than exponential
- The saturation of instabilities helps determine the resulting configuration of the system
- Numerical simulations are usually needed to investigate nonlinear growth and saturation

The Rayleigh-Taylor instability occurs when dense fluid sits on top of sparse fluid



- An example of interchange behavior with nonlinear dynamics and secondary instabilities
- Magnetized equivalent: light fluid → magnetic field

The Kelvin-Helmholtz instability results from velocity shear



- Results in characteristic Kelvin-Helmholtz vortices
- Above: Kelvin-Helmholtz instability in Saturn's atmosphere

Summary

- There are several different formulations for gauging MHD stability
- The initial value formulation provides the most information but typically requires the most effort
- The normal mode formulation is more amenable to linear analysis but not nonlinear analysis
- The variational approach retains information about stability and provides limits on the growth rate
- The energy principle is the most elegant method for determining stability and provides insight into the mechanisms behind instabilities, but does not help us determine the growth rate