Reduced MHD

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Astronomy 253: Plasma Astrophysics

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These lecture notes are largely based on *Lectures in Magnetohydrodynamics* by Dalton Schnack and *Nonlinear Magnetohydrodynamics* by Dieter Biskamp.

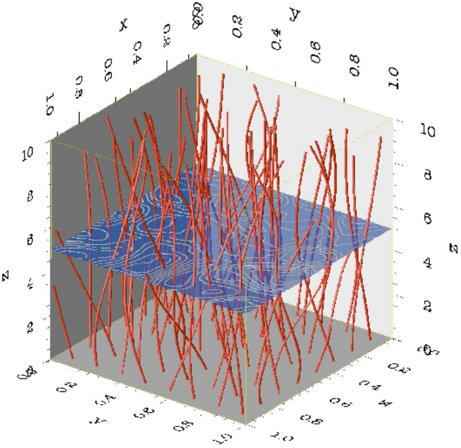
Outline

- Reduced MHD
  - A set of equations to describe the evolution of magnetic fields that are almost potential
- Parker's Problem
  - A model for magnetic topology evolution and energy release in coronal loops

Reduced MHD

- There are many situations where the magnetic field is almost uniform and unidirectional: $\mathbf{B} \approx B_z \hat{\mathbf{z}}$
  - Loops in the solar atmosphere
  - Magnetic clouds/flux ropes in the solar wind
  - Tokamaks and other magnetically confined fusion devices
  - Turbulence and transport in strongly magnetized plasmas
- Variations along $\mathbf{B}$ are rapidly smoothed out by parallel dynamics
- The field is *almost potential*
- The reduced MHD approximation allows us to simplify and avoid having to solve the full MHD equations in these situations

Reduced MHD is applicable when the magnetic field is almost uniform



- Note the vertical scale is smooshed!

The reduced MHD ordering

- We define the magnetic field to be

$$\mathbf{B} = \mathbf{B}_\perp + B_z \hat{\mathbf{z}} \tag{1}$$

- The small parameter ε is defined by

$$\frac{B_\perp}{B_z} \sim \varepsilon \ll 1 \tag{2}$$

- Each term in MHD will be ordered as some power of ε
- We will keep only the lowest powers of ε

Energetics of perpendicular dynamics

- Assume the perpendicular dynamics are in approximate energy equipartition
$$\frac{\rho V_\perp^2}{2} \sim \frac{p}{\gamma - 1} \sim \frac{B_\perp^2}{8\pi} \tag{3}$$
- Since $B_\perp \sim \varepsilon$ we then have
$$V_\perp \sim \varepsilon \tag{4}$$
$$p \sim \varepsilon^2 \tag{5}$$

  Because $p \sim \varepsilon^2$, pressure dynamics are neglected in this ordering
- Assume approximate force balance in parallel direction so that we can choose
$$V_z = 0 \tag{6}$$

Orderings of differential operators and resistivity

- Most of the spatial structure is in the plane perpendicular to the field so we use the orderings

$$\nabla_\perp \sim 1 \tag{7}$$
$$\frac{\partial}{\partial z} \sim \varepsilon \tag{8}$$

- We will consider situations where resistivity is small, but not too small:

$$D_\eta \sim \varepsilon \tag{9}$$

What is the order of variations in B_z?

- Write B_z as a sum of a constant, B_{z0}, and a varying part, $\tilde{B}_z$:

$$B_z \equiv B_{z0} + \tilde{B}_z \tag{10}$$

- Pressure balance in the z direction is given by

$$\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left(\frac{B_z^2}{8\pi}\right) = 0 \tag{11}$$

$$\Rightarrow \frac{\partial p}{\partial z} + \frac{B_{z0}}{4\pi}\frac{\partial \tilde{B}_z}{\partial z} = 0 \tag{12}$$

- Because $B_{z0} \sim 1$ and $p \sim \varepsilon^2$, this expression requires that

$$\tilde{B}_z \sim \varepsilon^2 \tag{13}$$

so that we can order variations in B_z out of the system

$$B_z = B_{z0} + \mathcal{O}\left(\varepsilon^2\right) \tag{14}$$

The reduced MHD ordering

- The reduced MHD ordering is

$$V_z = 0 \tag{15}$$

$$\nabla_\perp \sim 1, \qquad \frac{\partial}{\partial z} \sim \varepsilon, \tag{16}$$

$$D_\eta \sim \varepsilon, \qquad V_\perp \sim \varepsilon \tag{17}$$

$$p \sim \varepsilon^2, \qquad \tilde{B}_z \sim \varepsilon^2 \tag{18}$$

- Next, we move along with the derivation!
  - We are only sketching out the derivation
  - For a full derivation, see §2.5 of *Nonlinear Magnetohydrodynamics* by D. Biskamp or §13 of *Lectures in Magnetohydrodynamics* by D. Schnack

Ampere's law

- Evaluate Ampere's law and keep track of the order of each term

$$
\begin{aligned}
\frac{4\pi}{c}\mathbf{J} &= \nabla \times \mathbf{B} && (19)\\
&= \left(\underbrace{\nabla_\perp}_{\mathcal{O}(1)} + \underbrace{\hat{\mathbf{z}}\frac{\partial}{\partial z}}_{\mathcal{O}(\varepsilon)} \right) \times (\underbrace{\mathbf{B}_\perp}_{\mathcal{O}(\varepsilon)} + B_{z0}\hat{\mathbf{z}}) && (20)\\
&= \underbrace{\nabla_\perp \times \mathbf{B}_\perp}_{\mathcal{O}(\varepsilon)} + \mathcal{O}\left(\varepsilon^2\right) && (21)
\end{aligned}
$$

- The ordering is

$$
\begin{aligned}
\frac{4\pi}{c}J_z &\sim \varepsilon && (22)\\
\frac{4\pi}{c}\mathbf{J}_\perp &\sim \varepsilon^2 && (23)
\end{aligned}
$$

Define the $\mathbf{B}_\perp$ in terms of a flux function, ψ

- The magnetic field is given by

$$\mathbf{B} \equiv \hat{\mathbf{z}} \times \nabla\psi + B_{z0}\hat{\mathbf{z}} \tag{24}$$

  Here, $\psi = -A_z$ and we choose $\mathbf{A}_\perp = 0$
- The divergence constraint is automatically satisfied
- Now plug in the result from Ampere's law

$$\begin{aligned} \frac{4\pi}{c}\mathbf{J} &= \nabla \times (\hat{\mathbf{z}} \times \nabla\psi) & (25) \\ &= \hat{\mathbf{z}}\nabla^2\psi + \mathcal{O}\left(\varepsilon^2\right) & (26) \end{aligned}$$

- The current density is given by

$$\frac{4\pi}{c}J_z = \nabla^2\psi \tag{27}$$

Move on to Faraday's law and the resistive Ohm's law

- Put $\mathbf{B} \equiv \hat{\mathbf{z}} \times \nabla\psi + B_{z0}\hat{\mathbf{z}}$ in Faraday's law with Ohm's law

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \tag{28}$$

$$\frac{\partial}{\partial t}(\hat{\mathbf{z}} \times \nabla\psi) = -\nabla \times (-\mathbf{V} \times \mathbf{B} + \eta c\mathbf{J}) \tag{29}$$

Vector identity: $\nabla \times (\psi\hat{\mathbf{z}}) = \psi\nabla \times \hat{\mathbf{z}} - \hat{\mathbf{z}} \times \nabla\psi$

$$\nabla \times \frac{\partial}{\partial t}(\psi\hat{\mathbf{z}}) = \nabla \times (-\mathbf{V} \times \mathbf{B} + \eta c\mathbf{J}) \tag{30}$$

By uncurling we must include a scalar potential, χ

$$\hat{\mathbf{z}}\frac{\partial\psi}{\partial t} = -\mathbf{V} \times \mathbf{B} + \eta c\mathbf{J} + \nabla\chi \tag{31}$$

Now look at the parallel and perpendicular components

- The induction equation becomes

$$\frac{\partial\psi}{\partial t} = -\hat{\mathbf{z}}\cdot\mathbf{V}\times\mathbf{B} + \eta c J_z + \frac{\partial\chi}{\partial z} \tag{32}$$

$$0 = -(\mathbf{V}\times\mathbf{B})_\perp + \nabla_\perp\chi \tag{33}$$

- Since $V_z = 0$ we can write

$$\mathbf{V}\times\mathbf{B} = \underbrace{\mathbf{V}_\perp\times\mathbf{B}_\perp}_{\parallel \text{ to } \mathbf{B}} + \underbrace{B_{z0}\mathbf{V}_\perp\times\hat{\mathbf{z}}}_{\perp \text{ to } \mathbf{B}} \tag{34}$$

Introduce a stream function, ϕ

- The stream function ϕ describes the in-plane flow

$$\mathbf{V}_\perp = \hat{\mathbf{z}} \times \nabla\phi \tag{35}$$

- The in-plane flow is parallel to contours of constant ϕ
- The in-plane flow is incompressible:

$$\begin{aligned} \nabla \cdot \mathbf{V}_\perp &= \nabla \cdot (\hat{\mathbf{z}} \times \nabla\phi) \\ &= \nabla \cdot (\nabla \times \hat{\mathbf{z}}) - \hat{\mathbf{z}} \cdot (\nabla \times \nabla\phi) \\ &= 0 \end{aligned} \tag{36}$$

so we can choose a constant density, ρ_0 (often set to 1)

The parallel component of the induction equation

- After some vector manipulation, the parallel component of the induction equation becomes

$$\frac{\partial \psi}{\partial t} + \mathbf{V}_{\perp} \cdot \nabla \psi = D_{\eta} \nabla^2 \psi - B_{z0} \frac{\partial \phi}{\partial z} \tag{37}$$

  This relates the flux function ψ with the stream function ϕ and describes the evolution of the magnetic field

We still need an equation for the evolution of the flow

- Take the $\hat{\mathbf{z}}$ component of the curl of the momentum equation

$$\rho_0 \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{J} \times \mathbf{B} \tag{38}$$

$$\hat{\mathbf{z}} \cdot \rho_0 \left[\frac{\partial}{\partial t} \nabla \times \mathbf{V} + \nabla \times (\mathbf{V} \cdot \nabla \mathbf{V}) \right] = \hat{\mathbf{z}} \cdot \nabla \times (\mathbf{J} \times \mathbf{B}) \tag{39}$$

- Define the vorticity

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{V} \tag{40}$$

- The $\hat{\mathbf{z}}$ component of vorticity is found by

$$\omega_z = \nabla_\perp^2 \phi \tag{41}$$

The momentum equation yields the second equation of reduced MHD

- After some manipulation we arrive at

$$\rho_0 \left(\frac{\partial \omega_z}{\partial t} + \mathbf{V}_\perp \cdot \nabla \omega_z \right) = \mathbf{B} \cdot \nabla J_z \tag{42}$$

  This equation shows how the vorticity evolves in response to the plasma flow and Lorentz forces

- Here we used vector identities and dropped terms that were $\mathcal{O}\left(\varepsilon^2\right)$ or $\mathcal{O}\left(\varepsilon^3\right)$

The full equations of reduced MHD

- Reduced MHD is given by

$$\frac{\partial \psi}{\partial t} + \mathbf{V}_{\perp} \cdot \nabla \psi = D_{\eta} \nabla^2 \psi - B_{z0} \frac{\partial \phi}{\partial z} \tag{43}$$

$$\rho_0 \left(\frac{\partial \omega_z}{\partial t} + \mathbf{V}_{\perp} \cdot \nabla \omega_z \right) = \mathbf{B} \cdot \nabla \left(J_z \right) \tag{44}$$

with

$$\omega_z = \nabla_{\perp}^2 \phi \tag{45}$$

$$\frac{4\pi}{c} J_z = \nabla^2 \psi \tag{46}$$

$$\mathbf{V}_{\perp} = \hat{\mathbf{z}} \times \nabla \phi \tag{47}$$

- There are six equations for six unknowns: ψ, ω, ϕ, J_z, $\mathbf{V}_{\perp}$
- The equations for vorticity ω and J_z are Poisson-type

Takeaway points for reduced MHD

- Vector quantities are reduced to scalar functions: ψ and ω_z
- There are no parallel dynamics since these fast time scales have been ordered out of the problem
- Pressure dynamics are not included because they are $\mathcal{O}(\varepsilon^2)$
- Numerical solutions of reduced MHD are more efficient than solving the full equations of MHD
- Reduced MHD sometimes allows analytical progress that is not practical in full MHD

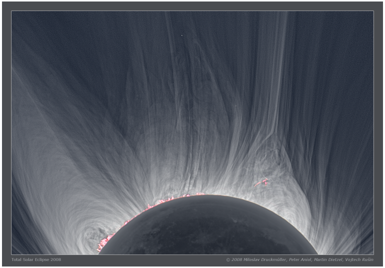
Extended Reduced MHD?

- It is possible to include pressure dynamics and compressibility in reduced MHD
  - Use $p \sim \varepsilon$ rather than $p \sim \varepsilon^2$
  - Useful for describing pressure-driven instabilities
- One can also include the Hall term
  - Useful to derive properties of some instabilities
- But the more terms you keep, the less of a simplification reduced MHD is!
- Numerical solution of full MHD is much more practical now than in the past
- Analytical solutions of full MHD are just as difficult
- Reduced MHD and its extensions still have important applications

There are multiple orderings in plasma physics to describe various phenomena

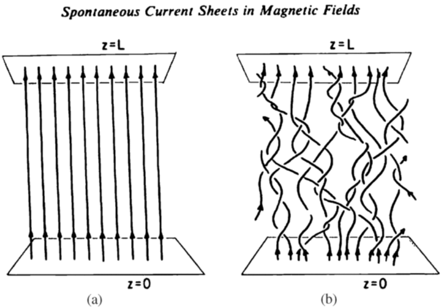
- Hall MHD ordering
  - Fast flows; relatively high frequencies
  - Hall term retained in Ohm's law
  - Applicable to some highly non-equilibrium situations
- MHD ordering
  - Flows comparable to ion thermal speed; low frequencies
  - Applicable to some situations that are not too far from equilibrium
- Drift ordering
  - Slow flows; very low frequencies
  - Particle drift velocities are comparable to flow velocities
  - Applicable to situations very close to equilibrium

Application of reduced MHD: heating of the solar corona



- The solar corona has $T \sim 10^6$ K even though the photosphere has $T \sim 6000$
- Only the magnetic field has enough energy to heat the plasma
- The two main proposed mechanisms for coronal heating are
  - Nanoflares: small, numerous reconnection events
  - Wave heating: dissipation of MHD waves
- Reduced MHD can be applied to nanoflare heating models through *Parker's problem*

Parker's conjecture



- Start with a uniform magnetic field between two endplates
  - This represents a stretched out/straightened coronal loop
- Mix up the footpoints (e.g., through flows on the endplates)
- Parker conjectured that it will be inevitable for current singularities to form where $\mathbf{J} \to \infty$ if $\eta = 0$

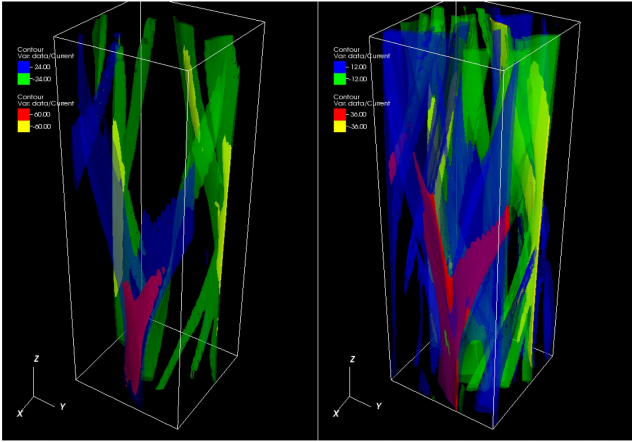
Parker's conjecture

- This is a difficult problem analytically
  - Parker's conjecture has not been proved
- This is a difficult problem numerically
  - Very difficult to simulate infinitely thin structures
- This is a difficult problem observationally
  - Magnetic fields very difficult to observe in the corona
  - Energy dissipation and heating occurs on tiny scales!

Parker's conjecture

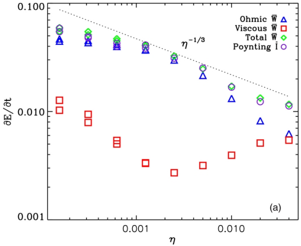
- Where does the energy go?
  - Source is kinetic energy of photospheric motions
  - The kinetic energy is stored as magnetic energy
  - Magnetic energy is dissipated into thermal energy
- Parker predicted quasi-static evolution between force-free states
- Parker's problem is not purely academic
  - There are real implications for how magnetic energy is dissipated in stellar coronae
  - This problem helps us understand how small-scale magnetic reconnection occurs in coronal loops

Reduced MHD simulations show the formation of current filaments (Ng et al. 2012)



- Flows on the endplates twist up the field
- Resistive dissipation releases stored magnetic energy

How does heating depend on Lundquist number?



- As $\eta \rightarrow 0$, the heating rate starts to plateau
- Does the heating rate become independent of Lundquist number as $S \rightarrow \infty$?
- Resistive heating is given by

$$Q_{tot} = \int_{\mathcal{V}} \eta J^2 \, \mathrm{d}\mathcal{V} \tag{48}$$

As η becomes smaller, the peak current increases and encompassing volume decreases

Taylor relaxation

- Taylor hypothesized that relaxation occurs by the magnetic field finding its minimum energy state while conserving helicity
- The final state will be force-free with constant α
  - A linear force-free field!
- In general, it is not possible to relax to a linear force-free field without changing the magnetic topology
  - Resistive diffusion or reconnection is required
  - This dissipation heats the plasma

Summary

- Reduced MHD is applicable in situations where the magnetic field is almost uniform
  - Provides many useful simplifications
- Reduced MHD puts the equations in terms of a flux function ψ and a stream function ϕ
- Investigations of Parker's conjecture provide insight into the nanoflare mechanism of coronal heating

Diagnosing Astrophysical Magnetic Fields

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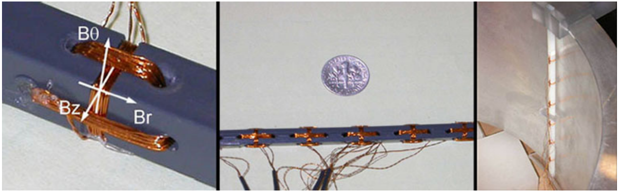
These lecture notes are largely based on Widrow (2002).

Outline

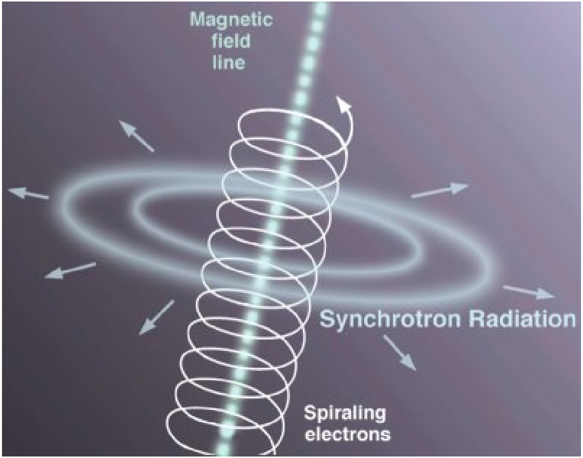
- In Situ Probes
- Synchrotron Radiation
- Faraday Rotation
- Zeeman Splitting
- Optical Polarization of Starlight

In situ probes



- In laboratory and near-Earth space plasmas, magnetic probes can measure $\partial \mathbf{B}/\partial t$ at a location in space
- Used to directly measure the magnetic field in magnetospheric and interplanetary plasmas (e.g., *ACE*, *Wind*, *STEREO*, *Cluster*, *MMS*)
- In situ probes in the laboratory can be used at low to medium temperatures (e.g., for the edges of tokamaks but not the hot core plasmas)

Synchrotron emission results from relativistic electrons spiraling along magnetic field lines



Synchrotron radiation provides information about $B_\perp$ and energetic electrons along the line of sight (LOS)

- Synchrotron emissivity is proportional to

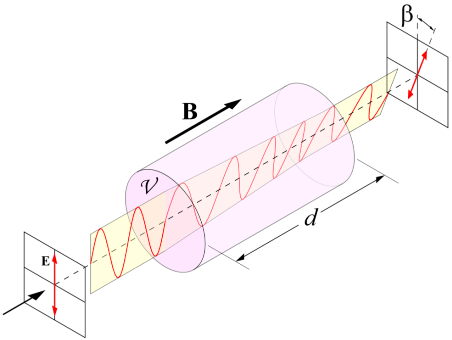
$$j_\nu \propto n_{e0} \nu^{(1-\gamma)/2} B_\perp^{(1+\gamma)/2} \tag{49}$$

where γ is the power law index for the electrons

$$n_e(E)\mathrm{d}E = n_{e0} \left(\frac{E}{E_0} \right)^{-\gamma} \mathrm{d}E \tag{50}$$

- Usually must make assumptions about the energetic particle population
- The degree of polarization depends on the uniformity of **B**

Faraday rotation provides information about $B_{\parallel}$ and n_e along the line of sight



- When polarized light propagates through a magnetized region with free electrons, the left- and right-circularly polarized components travel with different phase velocities

Faraday rotation

- Requires a polarized background source (often AGN)
- There is a rotation of the electric field vector:

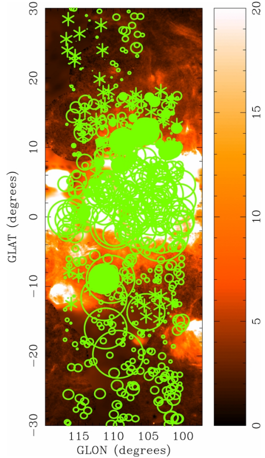
$$\Delta\varphi \propto \lambda^2 \int_0^L n_e(l) B_{\parallel}(l) \mathrm{d}l \tag{51}$$

- Hard to disentangle variations in n_e and $B_{\parallel}$ from each other, so we usually think in terms of a rotation measure (RM)

$$\Delta\varphi = \mathrm{RM}\,\lambda^2 \tag{52}$$

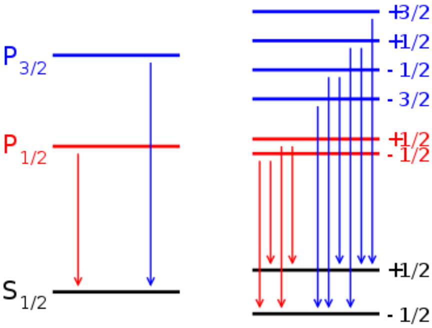
- Usually need three wavelengths to avoid the $n\pi$ ambiguity
- There are contributions to RM from the source itself, the IGM, and the ISM
- Faraday rotation is used to measure $\mathbf{B}_{\parallel}$ in laser produced plasmas (diagnostics for the Biermann battery!)

Example: Faraday rotation observations of the galactic Halo (Mao et al. 2012)



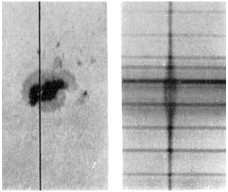
- The Milky Way could possess spiral like halo magnetic fields

Zeeman splitting



- Applying a magnetic field splits spectral lines into multiple components with a separation that is $\propto B$
- Energy levels that used to be degenerate have their degeneracy broken when there is a preferred direction associated with $\mathbf{B}$

The Zeeman effect diagnoses the total magnetic field at the source of the emission



- Hale used Zeeman splitting to show that sunspots are highly magnetized
- The Zeeman effect is weak unless the magnetic field is strong
- The weak shifts can be observed especially when the temperature and turbulent velocities are low
  - More like abnormal broadening than line splitting
- In the solar photosphere, can use various OIR lines
- For neutral hydrogen, can use the 21 cm line
- In molecular clouds, can use the 18 cm OH line

Polarization of optical starlight results from dust grain alignment with the magnetic field

- For prolate dust grains, one of the short axes will align with the magnetic field
- The dust will then preferentially absorb light polarized along its long axis
- The transmitted radiation will have a polarization direction parallel to **B**
- Provides information on $B_\perp$'s direction with a 180° ambiguity
- Difficulties include:
  - Need a lot of extinction for significant polarization to develop
  - Requires accurate knowledge of dust grain alignment processes
  - Scattering can also polarize starlight

From the review by Beck & Wielebinski (2013)

Field components are their observational signatures

Field component	Notation	Property	Observational signature
Total field	B	3D	Total synchrotron intensity, corrected for inclination
Total field in sky plane	$B_{\perp}$	2D	Total synchrotron intensity
Turbulent field in sky plane	$B_{turb,\perp}$	2D	Unpolarized synchrotron intensity, beam depolarization, Faraday depolarization
Turbulent field along line of sight	$B_{turb,\parallel}$	1D	Faraday depolarization
Ordered field perpendicular to the line of sight	$B_{ord,\perp}{}^2 = B_{an,\perp}{}^2 + B_{reg,\perp}{}^2$	2D	Polarized synchrotron intensity, optical polarization
Anisotropic field perpendicular to the line of sight	$B_{an,\perp}$	2D	Polarized synchrotron intensity, optical polarization
Regular field perpendicular to the line of sight	$B_{reg,\perp}$	2D	Polarized synchrotron intensity, optical polarization
Regular field along line of sight	$B_{reg,\parallel}$	1D	Faraday rotation and depolarization, Zeeman effect

Note that anisotropic fields and regular fields perpendicular to the line of sight cannot be distinguished

Summary

- In situ probes can be used in space and some laboratory plasmas to measure $\partial \mathbf{B}/\partial t$ directly
- Synchrotron radiation provides information on energetic electrons and $B_\perp$ along the line of sight
- Faraday rotation provides information on $B_{||}$ and n_e along the line of sight
  - Most important diagnostic for galactic/extragalactic $\mathbf{B}$
- The Zeeman effect is the most direct diagnostic of magnetic fields but is very difficult to detect
- Polarization of optical starlight results from alignment of dust grains by the magnetic field