

## THE MASS-RADIUS RELATION FOR COLD SPHERES OF LOW MASS\*

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Received 1969 March 14

### ABSTRACT

The relationship between mass and radius for zero-temperature spheres is determined for each of a number of chemical elements by using a previously derived equation of state and numerical integration. The maximum radius of a cold sphere is thus found as a function of chemical composition, and a semi-empirical formula for the mass-radius curve is derived.

For spherical bodies of given chemical composition and zero temperature, in hydrostatic equilibrium under their own gravitational forces, there is a unique relation between mass  $M$  and radius  $R$ . For very small masses, gravitational forces are small compared with electrostatic "solid state" forces, the density  $\rho$  has almost the zero-pressure value, and  $R \propto M^{1/3}$ . For larger masses (typical of small stars), the electrons are pressure-ionized, and electrostatic forces (effectively short-range because of charge neutrality) are small compared with Fermi pressure and with gravitation. In this regime  $R \propto M^{-1/3}$  (if relativistic effects are neglected; for classical white-dwarf theory, see e.g., Chandrasekhar 1939). For some intermediate value of the mass a finite maximum radius  $R_{\max}$  must therefore be reached. The value of this critical mass,  $M_{\text{cr}}$ , is at least of academic interest since it represents a natural dividing point between planets (and rocks) on the one hand, where gravity is a minor effect, and stars on the other hand, where Coulomb effects are small.

For hydrogen  $M_{\text{cr}}$  is close to the mass of Jupiter and has been calculated by DeMarcus (1958) in connection with some detailed models for the Jovian planets. For elements with larger atomic charge  $Z$  (in particular for iron,  $Z = 26$ )  $M_{\text{cr}}$  and  $R_{\max}$  have been estimated by Kothari (1938) and by Hamada and Salpeter (1961), using rather simple equations of state. In the present paper we present the results of numerical integrations which use an improved equation of state for zero-temperature matter, described elsewhere (Salpeter and Zapolsky 1967, hereinafter referred to as Paper I). This equation of state, derived from the Thomas-Fermi-Dirac differential equation including corrections for the electron correlation energy, is expected to be quite accurate at densities where at least one valence electron per atom is pressure-ionized. For masses near  $M_{\text{cr}}$  about half of all the  $Z$ -electrons are pressure-ionized, and our mass-radius relations should be quite accurate for  $Z \gg 1$ , but we also present results for hydrogen and helium which are only moderately accurate. In addition, we present a simple analytical semi-empirical formula for the mass-radius relation as a function of  $Z$ .

The familiar equations of hydrostatic equilibrium,

$$dP(r)/dr = -GM(r)r^{-2}\rho(r), \quad dM(r)/dr = 4\pi r^2\rho(r), \quad (1)$$

were integrated outward, starting with  $M(0) = 0$  and an assumed value  $P_c$  for the central pressure. The mass  $M$  and radius  $R$  are determined by the outer boundary condi-

\* Work supported in part by NASA Grant NGR-21-002-010 and NSF Grant GP8613.

tion  $P(R) = 0$ . For the equation of state we used a semi-empirical fitting formula with twenty-six parameters from Paper I, which gives density  $\rho$  as a function of pressure  $P$  for all  $Z$  between 1 and 92. All computations, using a fourth-order Runge-Kutta scheme, were carried out on an electronic computer. Our results for the mass-radius relation are given in Figure 1 for spheres of pure H,  ${}^4\text{He}$ ,  ${}^{12}\text{C}$ ,  ${}^{24}\text{Mg}$ , and  ${}^{56}\text{Fe}$ , and also for a H-He mixture, with  $X = 1 - Y$ , the abundance by weight of hydrogen. The maximum radius  $R_{\text{max}}$  and the corresponding mass  $M_{\text{cr}}$  are given in Table 1 for each composition.

The equation of state we used gives zero-pressure densities which are in error by factors up to 2 in the worst cases, since band-structure effects for the crystal lattice are omitted. These effects become small at pressures of a few times  $10^{12}$  dynes  $\text{cm}^{-2}$  (for a

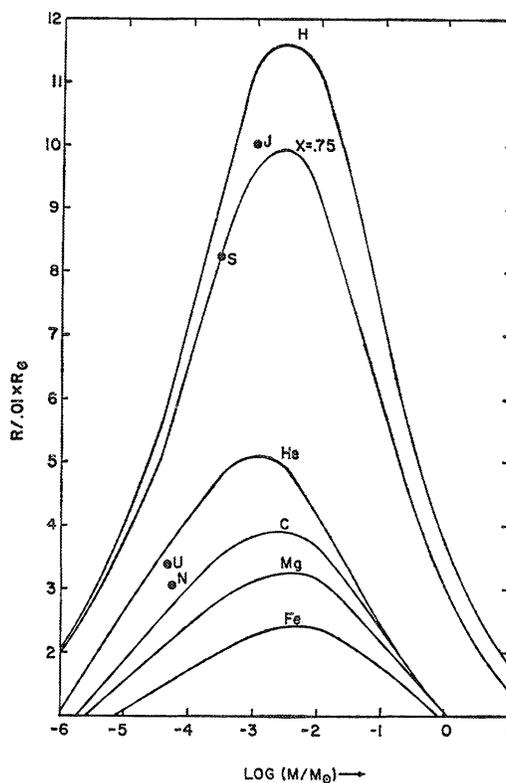


FIG. 1.—Mass-radius plot for homogeneous spheres of various chemical compositions. The points  $J$ ,  $S$ ,  $U$ ,  $N$  are the observed values for the Jovian planets.

TABLE 1

VALUES OF THE MAXIMUM RADIUS  $R_{\text{cr}}$  AND THE CORRESPONDING MASS  $M_{\text{cr}}$  AS A FUNCTION OF CHEMICAL COMPOSITION

| Element                     | $M_{\text{cr}}/0.001 M_{\odot}$ | $R_{\text{cr}}/0.01 R_{\odot}$ |
|-----------------------------|---------------------------------|--------------------------------|
| H . . . . .                 | 3.16                            | 11.71                          |
| $X=0.75$ $Y=0.25$ . . . . . | 2.63                            | 10.05                          |
| He . . . . .                | 1.12                            | 5.13                           |
| C . . . . .                 | 2.24                            | 3.94                           |
| Mg . . . . .                | 3.89                            | 3.28                           |
| Fe . . . . .                | 5.89                            | 2.44                           |

survey, see Zel'dovich and Raizer 1967), where our equation of state becomes accurate (Paper I). For the Fe models in Figure 1, material with  $P < 10^{12}$  dynes  $\text{cm}^{-2}$  contributes only 0.4 percent of the radius (and a negligible mass) at  $M_{\text{cr}}$ , and it still contributes only 3 percent of both the radius and mass for the model of lowest mass. Thus all our Fe models should be accurate. For the pure-H model with  $M_{\text{cr}}$ , the material with  $P < 10^{12}$  dynes  $\text{cm}^{-2}$  contributes 6 percent to the radius and 4 percent to the mass. These percentages decrease with increasing mass, and even our pure-H models should be fairly reliable for  $M \gtrsim M_{\text{cr}}$ , but not for much smaller masses. Our value of  $R_{\text{max}}$  for pure H is very close to that of DeMarcus (1958).

The observed masses and radii of the Jovian planets Jupiter, Saturn, Neptune, and Uranus are indicated as points in the  $(M, R)$ -plane of Figure 1. In Figure 2 we have plotted the radius, central pressure, and central density as a function of  $Y = 1 - X$

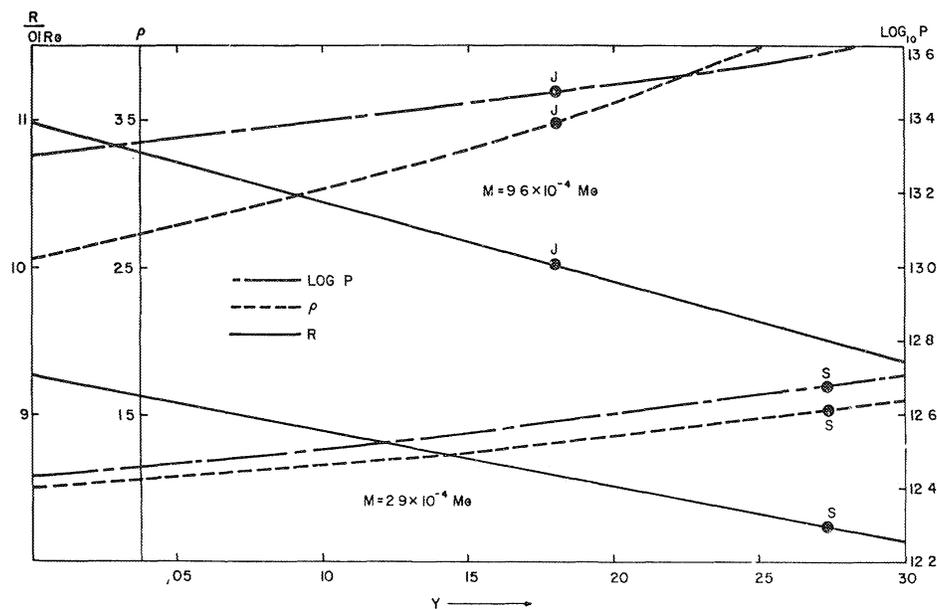


FIG. 2.—Radius  $R$ , central pressure  $P$  (dynes  $\text{cm}^{-2}$ ), and central density  $\rho$  ( $\text{g cm}^{-3}$ ) as a function of  $Y$  for spheres with masses fixed at the observed values for Jupiter and Saturn. Heavy points indicate the observed radii of these planets and the corresponding values of  $\rho$ ,  $P$ , and  $Y$  required by the equation of state in Paper I.

for two sets of models with masses held fixed at the observed values for Jupiter and Saturn. The  $(M, R)$ -values for Jupiter and Saturn can be fitted with our homogeneous H-He mixtures by models with  $X = 0.82$  and  $0.73$ ,  $\rho_c = 3.48$  and  $1.55$   $\text{g cm}^{-3}$ ,  $\log (P_c/\text{dynes cm}^{-2}) = 13.44$  and  $12.64$ , respectively. These models for Jupiter and Saturn are in reasonable agreement with the more accurate work of DeMarcus (1958) and Peebles (1964), who allowed nonuniform chemical composition in their models and fitted higher moments as well as  $M$  and  $R$ . It is clear from Figure 1 that an appreciable admixture of heavy elements is necessary to account for  $M$  and  $R$  of Neptune and Uranus, a conclusion in agreement with that of DeMarcus and Reynolds (1963).

Our results in Figure 1 can be summarized semiquantitatively as follows. It follows from dimensional arguments that the solutions to equation (1) are of the form

$$M = \beta R^3 \rho_c, \quad GM^2 = \delta R^4 P_c, \quad (2)$$

where  $\rho_c$  and  $P_c$  are central density and pressure, respectively. Here  $\beta$  and  $\delta$  are dimensionless numbers of order unity whose value depends on the degree of concentration of

mass in the model: for  $M \ll M_{\text{cr}}$  we have an almost incompressible model (polytrope of index  $n = 0$ ) with  $\beta = 4.19$ ,  $\delta = 8.38$ ; for  $M \gg M_{\text{cr}}$  (but with relativistic effects neglected) we have almost a perfect nonrelativistically degenerate Fermi gas which gives the ( $n = 1.5$ )-polytrope with  $\beta = .699$ ,  $\delta = 1.30$ ; for  $M \sim M_{\text{cr}}$  we have intermediate values of  $\beta$  and  $\delta$ .

A semiquantitative approximation to our equation of state can be written in the form

$$\rho \simeq \rho_0(\zeta + \phi)^3, \quad \zeta^5 \equiv P/P_0, \quad (3)$$

where

$$\begin{aligned} \rho_0 &= \frac{32}{3}\pi^{-3}AZH\alpha_0^{-3} = 3.88 \text{ g cm}^{-3}, \\ P_0 &= Z^{10/3} \frac{2^3(2\pi)^{1/3}}{15\pi^4} \left(\frac{e^2}{\hbar c}\right)^2 \frac{mc^2}{a_0^3} = 9.52 \times 10^{13} \times Z^{10/3} \text{ dynes cm}^{-2}, \\ \phi &= \frac{1}{20}3^{1/3} + \frac{1}{8}\left(\frac{3}{4}\pi^{-2}Z^{-2}\right)^{1/3}, \end{aligned} \quad (4)$$

with  $A$  and  $Z$  the atomic weight and charge of the element,  $m$  and  $e$  the mass and charge of the electron,  $\hbar$  and  $a_0$  Planck's constant/ $2\pi$  and the Bohr radius, respectively, and  $H$  the atomic mass unit. The relation for  $\rho$  in equation (3) is accurate for  $\zeta \ll 1$  (and leads to the polytrope of index  $n = 1.5$ ), but is of the correct order of magnitude for any  $\zeta$ . Using equations (2) and (3), one can derive expressions of  $M$  and  $R$  in terms of the central value  $\zeta_c$  of  $\zeta$ ,

$$\frac{R^2}{R_0^2} = \frac{\delta}{\beta^2} \frac{\zeta_c^5}{(\zeta_c + \phi)^6}, \quad \frac{M}{M_0} = \frac{\delta^{1.5}}{\beta^2} \frac{\delta_c^{7.5} \delta_c^{7.5}}{(\zeta_c + \phi)^6}, \quad (5)$$

where

$$\begin{aligned} R_0 &\equiv (P_0/G)^{1/2}\rho_0^{-1} = (Z/A)Z^{-1/3} \times 9.73 \times 10^9 \text{ cm}, \\ M_0 &\equiv (P_0/G)^{3/2}\rho_0^{-2} = (Z/A)^2Z \times 3.58 \times 10^{30} \text{ g} = 0.173(e^2/GH^2)^{3/2}(Z/A)^2ZH. \end{aligned} \quad (6)$$

If we keep  $\delta$  and  $\beta$  constant as we vary the parameter  $\zeta_c$ , we find for the maximum radius  $R_{\text{max}}$  and the corresponding values of  $M$  and  $\zeta_c$

$$\zeta_{c,\text{cr}} = 5\phi, \quad \frac{R_{\text{max}}^2}{R_0^2} = \frac{\delta}{\beta^2} \frac{5^5}{6^6\phi}, \quad \frac{M_{\text{cr}}}{M_0} = \frac{\delta^{1.5}}{\beta^2} \frac{5^{7.5}\phi^{1.5}}{6^6}. \quad (7)$$

Note that  $M_{\text{cr}}$  and  $M_0$  are of the order of  $(Z\alpha^{3/2})$  times the Chandrasekhar limiting mass, where  $\alpha \equiv e^2/\hbar c \simeq 1/137$  is the Sommerfeld fine-structure constant. Because of the smallness of  $\alpha$ , the mass range where Coulomb effects are important is well separated from the range where relativistic effects are important.

The parameter  $\zeta_c$  can also be eliminated from equation (5) to give a mass-radius relation

$$(M/M_0)^{1/3}R/R_0 = (\delta/\beta^{5/3})[1 - (R^3M_0/R_0^3M)^{1/3}\phi]^5. \quad (8)$$

Since equation (3) is a good approximation for  $\zeta \gg 1$ , equations (5) and (8) are accurate when  $\zeta \gg 1$  ( $M \gg M_0$ ), in which case we are dealing with the polytrope of index  $n = 1.5$  with  $\beta = 0.699$  and  $\beta^{5/3} = 0.42422\delta$ . For  $\zeta_c \lesssim 1$ ,  $M \lesssim M_{\text{cr}}$ , equations (5) and (8) with constant  $\beta$  and  $\delta$  (for a given value of  $A$ ) cannot be expected to be very accurate. We nevertheless found that equation (8) is tolerably close to the correct mass-radius relation in Figure 1 if  $\beta$  and  $\delta$  are adjusted to

$$\beta = 3.562 + 5.634 Z^{-1/2}, \quad \delta = \beta^{5/3}/0.4242. \quad (9)$$

The value of the ratio  $\beta^{5/3}/\delta$  ensures that equation (8) is accurate for  $M \gg M_0$  and, with the formula for  $\beta$  in equation (9), we found that equation (8) gave values for  $R_{\text{max}}$

within 1 percent of the correct values in Table 1 for all cases (for the H-He mixture we used the straightforward arithmetic averages for  $A$  and for  $Z$  of all the nuclei, as suggested in Paper I). The corresponding values of  $M_{\text{cr}}$  are in error by up to 25 percent, but the maxima of the curves in Figure 1 are rather flat, and equation (8) gives a rather good fit for  $M \gtrsim M_{\text{cr}}$  (but not  $M \ll M_{\text{cr}}$ , since eq. [8] with eq. [9] does not give the correct low-pressure densities).

This work was supported in part by NASA grant NGR-21-002-010 with the University of Maryland and by NSF grant GP8613 with Cornell University. We are also indebted, for some hospitality during the course of this work, to the NASA Institute for Space Studies, New York, and the Institute for Theoretical Astronomy, Cambridge University.

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