0.1 GRAVITATIONAL LENSING

The chance alignment of a foreground object along the line of sight to a high redshift source could result in the magnification, distortion, and potentially splitting of the source image due the deflection of its light rays by the gravitational field of the foreground object. The probability for *gravitational lensing* grows with increasing source redshift, due to the increase in the path length of the source photons. Although the lensing probability is only of anecdotal significance of $< 1\%$ for sources at $z < 2$, its magnitude could rise by an order of magnitude and affect the statistics of bright sources during the epoch of reionization.

Assuming that the gravitational potential of the lens is non-relativistic $|\Phi/c^2| \ll 1$, the effect of spacetime curvature on the propagation of light rays is equivalent to the effect of an effective index of refraction n,

$$n = 1 - \frac{2}{c^2}\,\Phi. \tag{1}$$

This follows from the deviation imparted to the phase of the electromagnetic wave by the potential of the lens (relative to a flat spacetime metric). The lens potential Φ is negative and approaches zero at infinity. As in normal geometrical optics, a refractive index $n > 1$ implies that light travels slower than in vacuum. Thus, the effective speed of a ray of light in a gravitational field is

$$v = \frac{c}{n} \simeq c - \frac{2}{c}\,|\Phi|\ . \tag{2}$$

The total time delay Δt, so-called the Shapiro delay, is obtained by integrating over the light path from the observer to the source:

$$\Delta t = \int_{\rm source}^{\rm observer} \frac{2}{c^3}\,|\Phi|\,dl\ . \tag{3}$$

A light ray is defined as the normal to the phase front. Since Φ and hence the phase delay of the electromagnetic wave vary across the lens, a light ray will be deflected by the lens as in a prism. The deflection is the integral along the light path of the gradient of n perpendicular to the light path, i.e.

$$\vec{\hat{\alpha}} = -\int \vec{\nabla}_\perp n\,dl = \frac{2}{c^2}\int \vec{\nabla}_\perp \Phi\,dl\ . \tag{4}$$

In all cases of interest the deflection angle is very small. We can therefore simplify the computation of the deflection angle considerably if we integrate $\vec{\nabla}_\perp n$ not along the deflected ray, but along an unperturbed light ray with the same impact parameter (with multiple lenses, one takes the unperturbed ray from the source as the reference trajectory for calculating the deflection by the first lens, the deflected ray from the first lens as the reference unperturbed ray for calculating the deflection by the second lens, and so on).

The simplest lens is a point mass, M, with a Newtonian potential,

$$\Phi(b, z) = -\frac{GM}{(b^2 + z^2)^{1/2}}\ , \tag{5}$$

where b is the impact parameter of the unperturbed light ray, and z indicates distance along the unperturbed light ray from the point of closest approach. We therefore have

$$\vec{\nabla}_\perp \Phi(b,z) = \frac{GM\,\vec{b}}{(b^2+z^2)^{3/2}} \ , \tag{6}$$

where $\vec{b}$ is orthogonal to the unperturbed ray and points toward the point mass. Equation (6) then yields the deflection angle

$$\hat{\alpha} = \frac{2}{c^2}\int \vec{\nabla}_\perp \Phi\, dz = \frac{4GM}{c^2 b} \ . \tag{7}$$

Since the Schwarzschild radius is $R_{\rm Sch} = (2GM/c^2)$, the deflection angle is simply twice the inverse of the impact parameter in units of the Schwarzschild radius. As an example, the Schwarzschild radius of the Sun is 2.95 km, and the solar radius is 6.96×10^5 km. A light ray grazing the limb of the Sun is therefore deflected by an angle $8.4 \times 10^{-6}\ \mathrm{radians} = 1.''7$.

The deflection angle from more a complicated mass distribution can be treated as the sum over the deflection caused by the infinitesimal point mass elements that make the lens. Since the deflection occurs on a scale $\sim b$ which is typically much shorter than the distances between the observer and the lens or the lens and the source, the lens can be regarded as thin. The mass distribution of the lens can then be replaced by a mass sheet orthogonal to the line-of-sight, with a surface mass density

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\, dz \ , \tag{8}$$

where $\vec{\xi}$ is a two-dimensional vector in the lens plane. The deflection angle at position $\vec{\xi}$ is the sum of the deflections from all the mass elements in the plane:

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2}\int \frac{(\vec{\xi}-\vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi}-\vec{\xi}'|^2}\, d^2\xi' \ . \tag{9}$$

In general, the deflection angle is a two-component vector. In the special case of a circularly symmetric lens, the deflection angle points toward the center of symmetry and has an amplitude,

$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi} \ , \tag{10}$$

where ξ is the distance from the lens center and $M(\xi)$ is the mass enclosed within radius ξ,

$$M(\xi) = 2\pi \int_0^\xi \Sigma(\xi')\xi'\, d\xi' \ . \tag{11}$$

The basic lensing geometry is illustrated in Figure 1. A light ray from a source S is deflected by the angle $\vec{\hat{\alpha}}$ at the lens and reaches an observer O. The angle between some arbitrarily-chosen axis and the true source position is $\vec{\beta}$, and the angle between the same axis and the image I is $\vec{\theta}$. The angular diameter distances

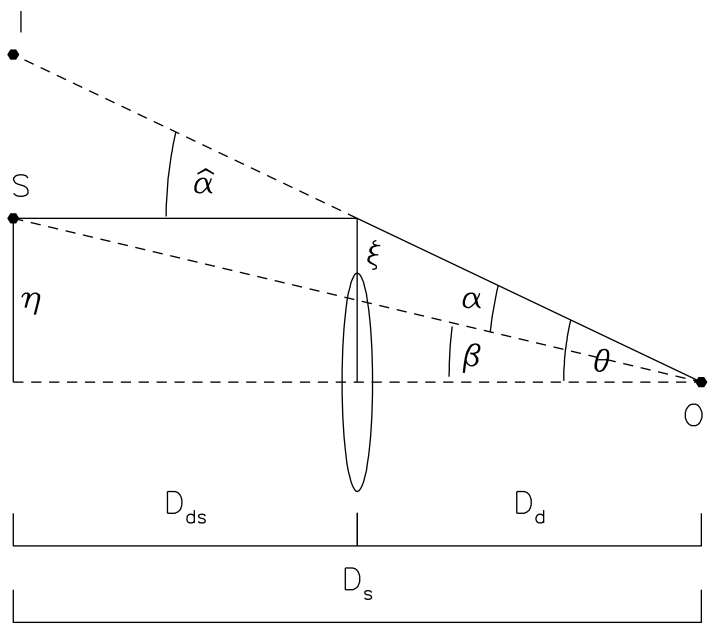

Figure 1 Geometry of gravitational lensing. The light ray propagates from the source S at transverse distance η from an arbitrary axis to the observer O, passing the lens at transverse distance ξ. It is deflected by an angle $\hat{\alpha}$. The angular separations of the source and the image from the axis as seen by the observer are β and θ, respectively. The distances between the observer and the source, the observer and the lens, and the lens and the source are $D_{\rm s}$, $D_{\rm d}$, and $D_{\rm ds}$, respectively.

between observer and lens, lens and source, and observer and source are denoted here as $D_{\rm d}$, $D_{\rm ds}$, and $D_{\rm s}$, respectively.

It is convenient to introduce the reduced deflection angle

$$\vec{\alpha} = \frac{D_{\rm ds}}{D_{\rm s}}\,\hat{\vec{\alpha}}\,. \tag{12}$$

The triangular geometry in Figure 1 implies that $\theta D_{\rm s} = \beta D_{\rm s} - \hat{\alpha} D_{\rm ds}$, so that the positions of the source and the image are related through the simple *lens equation*,

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})\,. \tag{13}$$

The nonlinear lens equation allows for multiple images $\vec{\theta}$ at a fixed source position $\vec{\beta}$. In a flat Universe, the comoving angular-size distances simply add up, with $D_{\rm ds}(1+z_s) = D_{\rm s}(1+z_s) - D_{\rm d}(1+z_d)$.

Because of the equivalence principle, the gravitational deflection is independent of photon wavelength. In addition, since the phase space density of photons must be conserved (Liouville's theorem), gravitational lensing preserves the surface brightness of the source and only changes its apparent surface area. The total flux received from a gravitationally lensed image of a source is therefore changed in proportion to the ratio between the solid angles of the image and the source. For a circularly symmetric lens, the magnification factor μ is given by

$$\mu = \frac{\theta}{\beta}\,\frac{d\theta}{d\beta}\,. \tag{14}$$

0.1.1 Special Examples of Lenses

0.1.1.1 Constant Surface Density

For a mass sheet with a constant surface density Σ, equation (10) implies a reduced deflection angle of,

$$\alpha(\theta) = \frac{D_{\rm ds}}{D_{\rm s}}\,\frac{4G}{c^2\xi}\,(\Sigma\pi\xi^2) = \frac{4\pi G\,\Sigma}{c^2}\,\frac{D_{\rm d}D_{\rm ds}}{D_{\rm s}}\,\theta\,, \tag{15}$$

where $\xi = D_{\rm d}\theta$. In this special case, the lens equation is linear with, $\beta \propto \theta$. Let us define a critical surface-mass density

$$\Sigma_{\rm cr} = \frac{c^2}{4\pi G}\,\frac{D_{\rm s}}{D_{\rm d}D_{\rm ds}} = 0.35\,{\rm g\,cm^{-2}}\left(\frac{D}{1\,{\rm Gpc}}\right)^{-1}\,, \tag{16}$$

where the effective distance D is defined through the following combination of distances

$$D = \frac{D_{\rm d}D_{\rm ds}}{D_{\rm s}}\,. \tag{17}$$

For a lens with $\Sigma = \Sigma_{\rm cr}$, the deflection angle is $\alpha(\theta) = \theta$, and so $\beta = 0$ for all θ. Such a lens focuses perfectly, with a single focal length. For a typical gravitational lens, however, light rays which pass the lens at different impact parameters cross at different distances behind the lens. Usually, lenses with $\Sigma > \Sigma_{\rm cr}$ somewhere in them, produce multiple images of the source.

0.1.1.2 Circularly Symmetric Lenses

For a circularly symmetric lens with an arbitrary mass profile, equations (10) and (12) give,

$$\beta(\theta) = \theta - \frac{D_{\rm ds}}{D_{\rm d} D_{\rm s}} \frac{4GM(\theta)}{c^2\,\theta} \ . \tag{18}$$

A source which lies exactly behind the center of symmetry of the lens ($\beta = 0$) is imaged as a ring. Substituting $\beta = 0$ in equation (18) yields the angular radius of the ring to be,

$$\theta_{\rm E} = \left[\frac{4GM(\theta_{\rm E})}{c^2} \frac{D_{\rm ds}}{D_{\rm d} D_{\rm s}}\right]^{1/2} \ . \tag{19}$$

This so-called *Einstein radius* defines the characteristic angular scale of lensed images: when multiple images are produced, the typical angular separation between the images images is $\sim 2\theta_{\rm E}$. Also, sources which are closer than $\sim \theta_{\rm E}$ in projection relative to the lens center, experience strong lensing in the sense that they are significantly magnified, whereas sources which are located well outside the Einstein ring are magnified very little. In many lens models, the Einstein ring also represents roughly the boundary between source positions that are multiply-imaged and those that are only singly-imaged. Interestingly, by comparing equations (16) and (19) we see that the mean surface mass density inside the Einstein radius is just the critical density $\Sigma_{\rm cr}$.

For lensing by a galaxy mass M at a cosmological distance D, the typical Einstein radius is

$$\theta_{\rm E} = (0.''4) \left(\frac{M}{10^{11}\, M_\odot}\right)^{1/2} \left(\frac{D}{5{\rm Gpc}}\right)^{-1/2} \ . \tag{20}$$

0.1.1.3 Point Mass

For a point mass M the lens equation has the form,

$$\beta = \theta - \frac{\theta_{\rm E}^2}{\theta} \ . \tag{21}$$

This equation has two solutions,

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_{\rm E}^2}\right) \ . \tag{22}$$

Any source is imaged twice by a point mass lens. The two images are on opposite sides of the source, with one image inside the Einstein ring and the other outside. As the source moves away from the lens (i.e. as β increases), one of the images approaches the lens and becomes very faint, while the other image approaches the true position of the source and asymptotes to its unlensed flux.

By substituting β from the lens equation (21) into equation (14), we obtain the magnifications of the two images,

$$\mu_\pm = \left[1 - \left(\frac{\theta_{\rm E}}{\theta_\pm}\right)^4\right]^{-1} = \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2} \ , \tag{23}$$

where u is the angular separation of the source from the point mass in units of the Einstein angle, $u = \beta\theta_{\rm E}^{-1}$. Since $\theta_- < \theta_{\rm E}$, $\mu_- < 0$, and so the magnification of the image which is inside the Einstein ring is negative implying that this image has its parity flipped with respect to the source. The net magnification of flux in the two images is obtained by adding the absolute magnifications,

$$\mu = |\mu_+| + |\mu_-| = \frac{u^2+2}{u\sqrt{u^2+4}}\ . \tag{24}$$

When the source lies on the Einstein radius, we have $\beta = \theta_{\rm E}$, $u = 1$, and the total magnification becomes

$$\mu = 1.17 + 0.17 = 1.34\ . \tag{25}$$

0.1.1.4 Singular Isothermal Sphere

A simple model for the mass distribution in a galaxy assumes that its collisionless particles (stars and dark matter) possess the same isotropic velocity dispersion everywhere. Surprisingly, this simple model appears to describe extremely well the dynamics of stars and gas in the cores of disk galaxies (whose rotation curve is roughly flat), as well the strong lensing properties of spheroidal galaxies.

We assume a spherically symmetric gravitational potential which confines the collisionless particles that produce it. We can associate an effective "pressure" with the momentum flux of these particles at a mass density ρ,

$$p = \rho\sigma_v^2, \tag{26}$$

where σ_v is the one-dimensional velocity dispersion of the particles, assumed to be constant across the galaxy. The equation of hydrostatic equilibrium (which is derived from the second moment of the collisionless Boltzmann equation) gives

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{GM(r)}{r^2}\ , \quad \frac{dM(r)}{dr} = 4\pi\, r^2\, \rho\ , \tag{27}$$

where $M(r)$ is the mass interior to radius r. A particularly simple solution of equations (26) through (27) is

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\,\frac{1}{r^2}\ . \tag{28}$$

This mass distribution is called the *singular isothermal sphere* (and will be abbreviated as SIS below). Since $\rho \propto r^{-2}$, the mass $M(r)$ increases $\propto r$, and therefore the rotational velocity of test particles in circular orbits in the gravitational potential is

$$V_{\rm c}^2(r) = \frac{G\,M(r)}{r} = 2\,\sigma_v^2 = \text{constant}\ . \tag{29}$$

As mentioned, the flat rotation curves of disk galaxies are naturally reproduced by this model.

By projecting the mass distribution along the line-of-sight, we obtain the surface mass density,

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G}\,\frac{1}{\xi}\ , \tag{30}$$

where ξ is the distance from the center of the two-dimensional profile. The reduced deflection angle from (10),

$$\hat{\alpha} = 4\pi \frac{\sigma_v^2}{c^2} = (1.''16) \left(\frac{\sigma_v}{200\,\mathrm{km\,s^{-1}}}\right)^2 , \tag{31}$$

is independent of ξ and points toward the center of the lens. The Einstein radius of the SIS follows from equation (19),

$$\theta_{\mathrm{E}} = 4\pi \frac{\sigma_v^2}{c^2} \frac{D_{\mathrm{ds}}}{D_{\mathrm{s}}} = \hat{\alpha} \frac{D_{\mathrm{ds}}}{D_{\mathrm{s}}} = \alpha\ . \tag{32}$$

Due to circular symmetry, the lens equation is one dimensional. Multiple images are obtained only if the source lies inside the Einstein ring. If $\beta < \theta_{\mathrm{E}}$, the lens equation has the two solutions

$$\theta_{\pm} = \beta \pm \theta_{\mathrm{E}}\ . \tag{33}$$

The images at $\theta_{\pm}$, the source, and the lens all lie on a straight line. Technically, a third image with zero flux is located at $\theta = 0$; this image acquires a finite flux if the divergent density at the center of the lens is replaced by a core region with a finite density.

The magnifications of the two images follow from equation (14),

$$\mu_{\pm} = \frac{\theta_{\pm}}{\beta} = 1 \pm \frac{\theta_{\mathrm{E}}}{\beta} = \left(1 \mp \frac{\theta_{\mathrm{E}}}{\theta_{\pm}}\right)^{-1} . \tag{34}$$

If the source lies outside the Einstein ring, i.e. if $\beta > \theta_{\mathrm{E}}$, there is only one image at $\theta = \theta_{+} = \beta + \theta_{\mathrm{E}}$.

0.1.2 Lensing Probability

A SIS lens has the simple property that the deflection angle $\hat{\alpha}$ is independent of the impact parameter of the light ray. The condition for multiple imaging (and hence strong lensing) is then that the source would lie inside the Einstein radius. The probability that a line-of-sight to a source at a redshift z_s passes within the cross-sectional area associated with the Einstein radius of SIS lenses $\pi\theta_E^2$ gives a lensing optical depth,

$$\tau(z_{\mathrm{s}}) = \frac{16\pi^3}{H_0} \int_0^{z_s} dz \frac{D^2(1+z)^2}{(\Omega_m(1+z)^3 + \Omega_\Lambda)^{1/2}} \int_0^{\infty} d\sigma_v \frac{dn}{d\sigma_v} \sigma_v^4, \tag{35}$$

where $(dn/d\sigma_v)d\sigma_v$ is the (redshift-dependent) comoving density of SIS halos with a one-dimensional velocity dispersion between σ_v and $\sigma_v + d\sigma_v$.

In calculating the probability of lensing it is important to allow for various selection effects. Lenses magnify the observed flux, and lift sources which are intrinsically too faint to be observed over the detection threshold. At the same time, lensing increases the solid angle within which sources are observed so that their number density in the sky is reduced. If there is a large reservoir of faint sources, the increase in source number due to the apparent brightening outweighs their spatial dilution, and the observed number of sources is increased due to lensing. This so-called magnification bias can substantially increase the probability of lensing

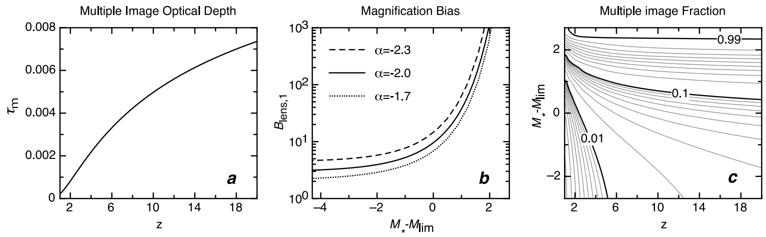

Figure 2 Probability for multiple imaging of high redshift galaxies by an unevolving population of SIS lenses. *Panel a*: lensing probability τ as a function of source redshift. *Panel b*: magnification bias as a function of the difference between the characteristic magnitude of a galaxy $M_\star$ (assuming a Schechter luminosity function) and the limiting survey magnitude $M_{\rm lim}$. Three values of the faint-end slope of the luminosity function (labeled by α here) are shown. *Panel c*: Contours of the fraction of multiply-imaged sources as a function of source redshift and $(M_\star - M_{\rm lim})$, assuming a faint end slope of -2. Figure credit: J. S. B. Wyithe, et al. Nature **469**, 7329 (2011)

for bright sources whose number-count function is steep. The magnification bias for sources at redshift z_s with luminosities between L and $L + dL$ is,

$$B(L) = \frac{1}{dn_s(L)/dL} \int_{\mu_{\rm min}}^{\mu_{\rm max}} \frac{d\mu}{\mu} \frac{dP}{d\mu} \frac{dn_s(L)}{dL}, \tag{36}$$

where $n_s(< L)$ is the density of sources with luminosity $< L$ and $dP/d\mu$ is the probability for magnification μ. For example, the brighter SIS image has a magnification distribution $(dP/d\mu) = 2(\mu - 1)^{-3}$ for $2 < \mu < \infty$.

A simplified model for the redshift evolution of SIS lenses is to use the mass function of dark matter halos that was derived in §**??** and identify $\sigma_v = V_{\rm c}/\sqrt{2}$ at the virial radius. Another simplified approach is to adopt the observed $(dn/d\sigma_v)$ at $z = 0$ and assume no evolution in the comoving density of lenses. The latter approach gives the approximate results shown in Figure .