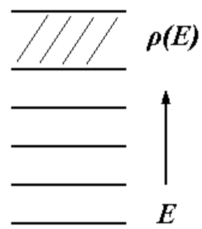
2. Blackbody Radiation, Boltzmann Statistics, Temperature, and Equilibrium

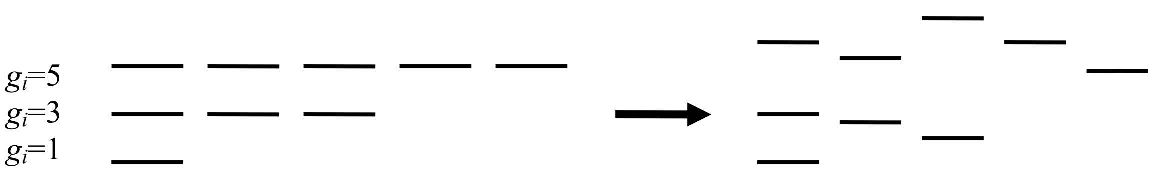
Penner, Chapters 1 and 2 has great details. A good statistical mechanics book (like **Davidson**, Chapters 7-12) is a good source for further information especially on statistical aspects.

Blackbody radiation, temperature, and thermodynamic equilibrium give a tightly coupled description of systems (atmospheres, volumes, surfaces) that obey Boltzmann statistics. They are important because of the compact descriptions of systems that they give when Boltzmann statistics apply, either approximately or nearly exactly. Fortunately, this is most of the time in the Earth's stratosphere and troposphere, and in other planetary atmospheres as long as the density is sufficient that collisions among atmospheric molecules, rather than photochemical and photophysical properties, determine the energy populations of the ensemble of molecules.

Consider a set of energy states an atom or molecule may occupy:



There are a number of discrete energy states and, above the *dissociation limit*, where the molecule is no longer bound, a continuum described by the density of states $\rho(E)$. Ignoring the continuum until later, the Boltzmann factor for a given state, i, at energy E_i, is $e^{-E_i/kT}$, where k is Boltzmann's constant and T is the temperature. We will also need the concept of *degeneracy* of states. This is nothing more than the realization that there are, in many or most cases, more than one distinct quantum state at each energy. For example, in the probably familiar case of the p-orbitals of an atom, there are three distinct states, where the lobes of the orbitals (each of which can house two electrons) are oriented along either the x- y- or z-axis. These orbitals are degenerate in energy. In the soon-to-be-familiar case of the rotational states of a diatomic molecule, a state with rotational angular momentum J has a degeneracy of $2J+1$. This degeneracy may be *broken* by an electric or magnetic field, if the molecule has either an electric or magnetic dipole moment, to separate the states in energy. g_i = the number of energy levels with energy E_i:



The degeneracy may be broken, *e.g.*, by $\vec{m}$ **or** $\vec{E}$.

For the present, we simply need to realize that degeneracy means having more than one state at a particular energy. We may then describe Boltzmann statistics:

The population in each state P_i is $\propto e^{-E_i/kT}$.
The population at each energy P_{E_i} is $\propto g_i e^{-E_i/kT}$.
The *partition function* $Q \equiv \sum_i g_i e^{-E_i/kT}$.
The extension to include continuum states is apparent:

$$Q \equiv \sum_{i=1}^{b} g_i e^{-E_i/kT} + \int_b^{\infty} \rho(E) e^{-E/kT} dE.$$

We will omit the continuum term from further discussions, as we will not normally need it, since the dissociation energies are so high that continuum states do not normally contribute to the populations of planetary atmospheres in near-equilibrium conditions. The partition function provides the normalization factor so that populations may now be given as:

$P_i = e^{-E_i/kT}/Q.$
The population at energy E_i, $P_{E_i} = g_i e^{-E_i/kT}/Q$.

The normalization by Q also makes the arbitrary choice of the zero of energy cancel out of the population statistics. *(See why? Try showing it as an exercise.)*

Now we may state some definitions a bit more firmly:

- A system at equilibrium is one where the populations of energy levels are described by Boltzmann statistics;
- A system at equilibrium may be described by a temperature and, conversely;
- Temperature is a characteristic of an equilibrium system. No equilibrium, no defined temperature;
- At equilibrium, the radiation is in equilibrium with the molecules, at the same temperature. It is described by the blackbody radiation law, which we shall meet shortly.

One often meets the expression *local thermodynamic equilibrium* (LTE), particularly in astrophysics and atmospheric science. This is simply a way of expressing that local behavior (say at a certain altitude in the atmosphere) is reasonably well described as being in equilibrium and characterized by a temperature, whereas on larger scales (as we know already from discussions of atmospheric structure) this cannot be the case, since the temperature varies.

An atmosphere may be stable even though it is not in equilibrium, depending on the boundary conditions of gravity and heating. Most are not in equilibrium (no planetary atmospheres are): Since the Earth's troposphere is heated from the bottom, and warmer air is more buoyant, there is a natural mixing as a counter effect to the thermodynamic temperature lapse (*tropos* is Greek for "to turn"). The stratosphere, on the other hand, is heated from the top, as we will see later. It is more *stratified*, and stable. LTE is not necessary for a system even locally to have a constant energy distribution. In regions that are not in LTE it is possible to be in a radiative *steady-state*, analogous to the chemical kinetic steady-state situation. As a very rough rule, LTE in a region of an atmosphere is established when there are ≥ 10 collisions per photochemical or reaction event. In the Earth's atmosphere, non-LTE conditions are normally encountered in the mesosphere and above, with mesospheric CO_2 as a common example.

Situations where a system is described by more than one temperature are frequently encountered. For example, in an astrophysical plasma, one may hear of a "radiation temperature" and a different "kinetic temperature." In this case the separate phases (radiation and matter) are reasonably well described by temperatures, but are not strongly coupled together through absorption and emission to establish equilibrium. Analogously, in laboratory spectroscopy, one frequently hears of separate rotational and vibrational temperatures produced by certain sample preparation techniques (*e.g.*, supersonic expansions in molecular beams).

Blackbody radiation

What is a blackbody (BB)?

- BB absorptance = 1 (optical thickness = ∞), reflectance = 0.
- BB completely characterized by a temperature *T*.
- BB is an ideal situation, but is important for emission and absorption spectroscopy, as we will see.
- BB emission of radiation given by Planck's law (below).
- BB emission is *Lambertian* ($\propto \cos\theta$, where θ is the angle normal to the surface, again, an approximation for real situations). Lambertian also means that reflection is diffuse (when there is reflection, *i.e.*, for a surface that is not a BB).
- A buffalo or a moose is a pretty good blackbody.

Planck's law: Blackbody radiancy $R_\sigma d\sigma = \dfrac{2\pi hc^2\sigma^3 d\sigma}{e^{hc\sigma/kT}-1}$ (erg s^{-1} cm^{-2}).

$R_\sigma d_\sigma$ is sometimes called $W_\sigma\, d\sigma$. σ is the wavenumber, (cm^{-1}), *h* is the Planck constant (6.62606957×10^{-27} erg s)[1], *c* the speed of light (2.99792458×10^{10} cm s^{-1}), and *k* the Boltzmann constant (1.3806488×10^{-16} erg ^{o}K^{-1}). This is blackbody emission from a surface, *not* the same as the BB radiation density, described in **Goody & Yung**. Note the cgs units, instead of Joules and meters.

The first radiation constant $c_1 \equiv 2\pi hc^2$ = 3.74177153×10^{-5} erg cm^{-2} s^{-1}. The second radiation constant $c_2 \equiv hc/k$ = 1.4387770 ^{o}K/cm^{-1}.

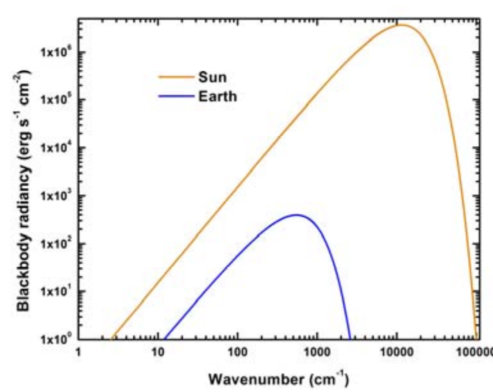


Figure 2.1 Emission per unit area is greater at every wavelength/frequency when the temperature is higher.

[1] I use COSPAR2010 values of the fundamental constants.

Often, excitation levels are given in °K rather than in cm^{-1}, for convenience in relating atomic and molecular physics to a particular temperature regime. The radiancy is thus:

$$R_\sigma d\sigma = \frac{c_1\sigma^3 d\sigma}{e^{c_2\sigma/T}-1}.$$

Homework (Assigned January 31, due February 9): Determine BB ***ster***radiancy (erg s^{-1} cm^{-2} sr^{-1}) by invoking Lambertian emission and integrating over solid angle.

In photons ($E = h\nu = hc\sigma$),

$$R_n(\sigma)d\sigma = \frac{2\pi c\sigma^2 d\sigma}{e^{c_2\sigma/T}-1} = \frac{1.883652\times10^{11}\sigma^2 d\sigma}{e^{c_2\sigma/T}-1}$$ (photons s^{-1} cm^{-2}).

Rayleigh-Jeans limit:

For $h\nu \ll kT (hc\sigma \ll kT)$, $R_\sigma \simeq$ linear with temperature:

$$\frac{2\pi hc^2\sigma^3 d\sigma}{e^{hc\sigma/kT}-1} \simeq 2\pi kTc\sigma^2 d\sigma.$$ This is in common use in radiofrequency and microwave work, especially in radioastronomy. This law was first discovered empirically, and then led to the predicted *ultraviolet catastrophe*. That is, emission versus wavenumber and total emission become infinite! It was known to be wrong (*c.f. Davidson*) but it was classically required. Quantum mechanics was required for the derivation of Planck's law.

Aside: *Antenna temperature*
Radioastronomers and aeronomers give power sources in this unit and talk about detectors, noise, and systems in this way – more soon.

More details

A blackbody is the most a surface at temperature *T* can emit. It can emit less:
Emissivity: $\varepsilon \le 1$.
Reflection coefficient: $R \le 1$.
Kirchoff's law: $\varepsilon + R = 1$.

Stefan-Boltzmann constant:
Integrating BB over σ gives 5.670373×10^{-5} erg cm^{-2} s^{-1} $°K^{-4}$ = 5.670373×10^{-8} W m^{-2} $°K^{-4}$.

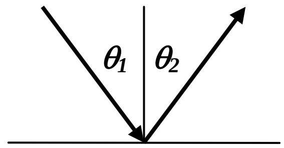
Wien's law:

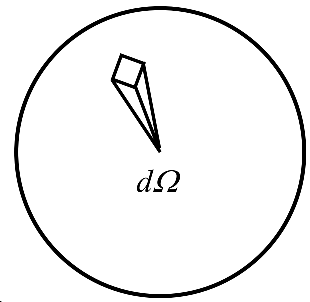
The maximum power per wavenumber occurs at σ_{max} (cm^{-1}) = 1.961008*T* (°K) (COSPAR 2006)
The maximum # of photons per wavenumber occurs at σ (n_{max}) (cm^{-1}) = 1.1076256*T* (°K) (COSPAR 2006)

The maximum power per μm occurs at λ = 2897.7721 / *T* (°K)
The maximum # of photons per μm occurs at λ = 3669.6986 / *T* (°K) (COSPAR 2006)

Bi-Directional Reflectance Distribution Function (BRDF)

Normal reflectance:





BRDF (θ_1, θ_2, φ) can be more general (φ is the azimuthal angle). This can be important in radiative scattering from surfaces.

Lambertian BRDF: $R \propto \cos\theta_2$. There is also a general polarization dependence (of which more later).

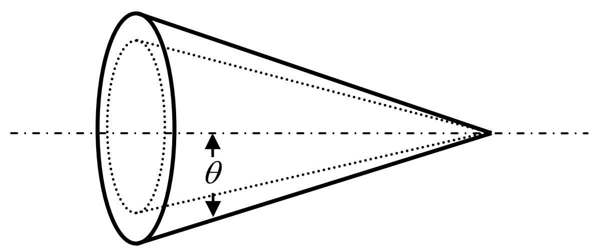
Some elements, for future use

Spherical geometry $\Rightarrow$ cones $\Rightarrow$ étendue

Spherical surface: $4\pi r^2$

Element of solid angle: $d\Omega = d\theta d\phi,\ \int\limits_{sphere} d\theta d\phi = 4\pi$

Area on a unit sphere: $\Omega = 2\pi\int_0^\theta \sin\theta' d\theta' = 2\pi(1-\cos\theta)$.

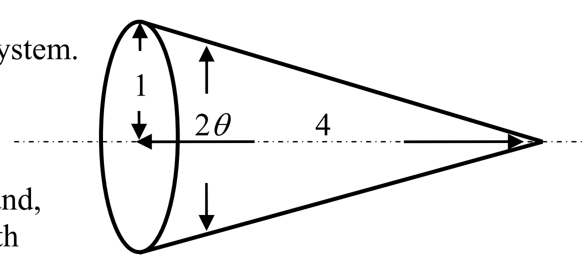


For $\theta = 90^\circ$ (*i.e.*, a plane), $\Omega = 2\pi$ steradians (sr).
For θ small (and in radians), $\Omega \approx \pi\,\theta^2$.

Étendue:
Put a (small) hole, of area *a*, in the tip of the cone above. Then É = *a* Ω. This is a "constant" of an optical system. It cannot be increased (although it can be decreased, by *vignetting*, or blocking part of the signal).

Ω large $\Rightarrow$ "fast" optical system; Ω small $\Rightarrow$ "slow" optical system.

f-number or f-stop = $1/2\tan\theta$ (this shows an *f/2* system)

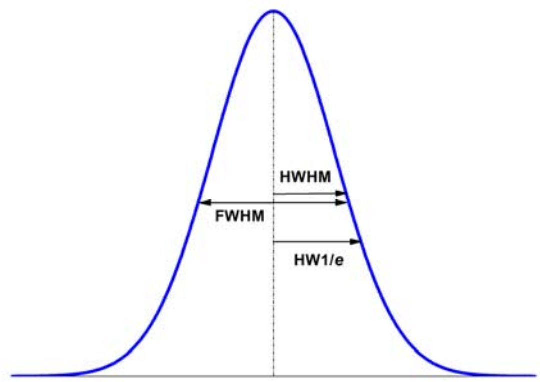


It is important to realize that atmospheric (and astronomical and, indeed, laboratory spectroscopic) measurements are made with instruments having properties described by their étendues. It is often a convenient approximation (one we shall employ frequently) to describe spectroscopic problems as plane-parallel. The angular situation is always lurking underneath.

Homework (Assigned January 31, due February 9): Construct an example where one observes an extended source (*e.g.*, a cloud) with an instrument having a given étendue. Show that the étendue is the same for the cloud observing you.

The Horiba Jobin Yvon Company has an excellent website giving a tutorial on the optics of spectroscopy: http://www.horiba.com/us/en/scientific/products/optics-tutorial/

3. Gaussians, noise, signal-to-noise ratio (S/N)



Horowitz and Hill, Chapter 7 has valuable discussions of noise types and sources.

A Gaussian line,

$$l_g(\sigma) = \frac{\pi^{-1/2}}{b_e} \exp-\left[\frac{(\sigma-\sigma_0)^2}{b_e^{\,2}}\right], \int_{-\infty}^{\infty} l_g(\sigma)\, d\sigma = 1.$$

where b_e is the *half-width at 1/e intensity* (hw1/e), to be compared later to the *half-width at half maximum (HWHM)* and the *full-width at half maximum (FWHM)*. I prefer using hw1/e to describe Gaussians and HWHM for *Lorentzian* lineshapes (of which more later).

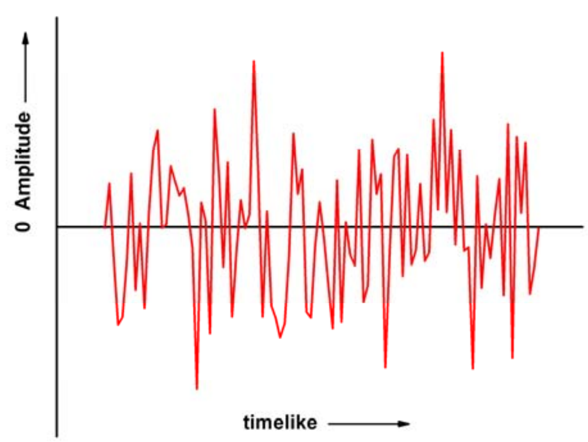
Gaussians widths add in *quadrature* (when convolving): $b_{total} = \sqrt{b_1^{\,2} + b_2^{\,2}}$ (Easy to show with the convolution theorem – *try it!*)

Signal-to-noise-ratio, S/N

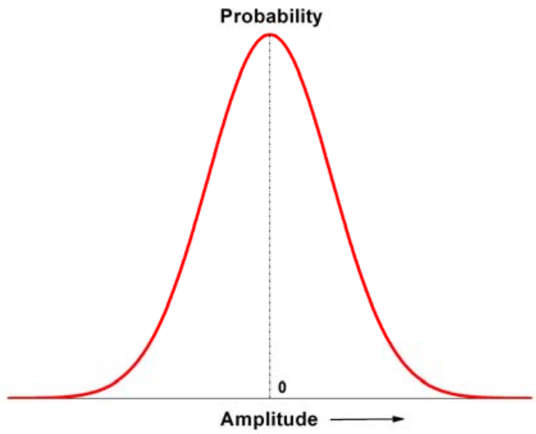
Signal: The signal of a system increases linearly with power. **Antenna temperature**, T_A: The signal (at a particular wavenumber, σ, or frequency, ν) is equivalent to the antenna being enclosed in a blackbody of temperature T. T_A of a line is usually defined for the line center.

Noise: We usually have (approximately) *band-limited white Gaussian noise*:

- Equal power per Hz (or cm^{-1}: a frequency unit)
- Gaussian distribution (±) of amplitudes



Measure at a given frequency:



Gaussian description of noise:
For noise, $\sigma_0 = 0, b_n = b_e / \sqrt{2}$
b_n = root-mean-square (RMS) noise = our noise for S/N purposes
Probability of amplitude

$$A, P_A = \frac{(2\pi)^{-1/2}}{b_n} \exp - \left[\frac{A^2}{2b_n^{\ 2}} \right]$$

Noise integrates up as $\sqrt{t}$ (because Gaussians add in quadrature), while signal integrates up as $t \Rightarrow$ S/N increases as $\sqrt{t}$.

Types of noise:

1. Noise components from the instruments (detector noise, readout noise, electronic noise) will generally be independent of the spectral intensity. They are generally (to a reasonable degree of fidelity) described as Gaussian white noise. In radio physics and astronomy noise, squares of noise sources are often described as *temperatures*, which add linearly to give a noise system *temperature*: Remember that, in the Rayleigh-Jeans limit, power is linearly proportional to temperature. Since noise increases as $\sqrt{\text{power}}$, again because sources add in quadrature, noise temperature sources add linearly.

2. A component to the whole system noise that is due to photon statistics, that is, to the fact that we are counting a discrete number of photons, *N*, is also proportional to $\sqrt{N}$ (proportional to $\sqrt{t}$ for linear integration). The S/N is thus proportional to $N / \sqrt{N} = \sqrt{N}.$ Where the spectrum is larger (say, at the peak of an emission line), the noise will be larger than at the trough, but the S/N signal will be lower. The margin of error (1 standard deviation, although almost never stated) usually given in political polls is $1/\sqrt{N}$, where *N* is the number of persons polled. This can result in less popular candidates having possibly negative approval ratings or likely voters! See where the problem arises?

Aside: The second noise source is described by Poisson statistics: Poisson statistics describes discrete events. From the ***Wikipedia*:**

In probability theory and statistics, the **Poisson distribution** *is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed* *period of time if these events occur with a known average rate and independently of the time since the last event. The Poisson distribution can also be used for the number of events in other specified intervals such as distance, area or volume ... The fluctuations about the mean value of events are denoted as* ***Poisson noise*** *or (particularly in electronics) as* ***shot noise****.*

Aside: A good noise generation program is often very useful. **noise.f90** is available at the class website. You may want to generate a noise spectrum with this program and test it to see how Gaussian the amplitude distribution is.

Back to system temperature (T_{sys}) and noise temperature (T_N). At low σ,

$$R(\sigma_0) = \frac{2\pi hc^2\sigma^3}{e^{c_2\sigma/T} - 1} \simeq 2\pi kTc\sigma^2 \equiv \text{ Rayleigh-Jeans } (RJ) \text{ limit.}$$

$$(2\pi hc^2 = 3.74177118\times10^{-5}; 2\pi kc = 2.6006643\times10^{-5})$$

T_{sys} and T_N are defined for 1 second integration time ($\propto T\times t^{-1/2}$).

Class problem: The cosmic microwave background (CMB) is $\simeq$ a BB @ 2.75 K; Measure it at 10 cm wavelength (= 0.1 cm^{-1} = 3GHz).

1. Are we @ $h\nu \ll kT$? (and therefore in the Rayleigh-Jeans limit?)
2. Determine the T_{sys} versus integration time to make a 1% measurement.

1. $R_\sigma = 6.966\times10^{-7}; \quad RJ = 7.152\times10^{-7}$

   $h\nu / kT = hc\sigma / kT = 0.0523$

   $h\nu < kT$ but maybe not $\ll kT$. Let's proceed anyway, as an example.

   $T_{sys} \times t^{-1/2} = 0.01\times 2.75K = 0.0275$

   $t^{1/2} = T_{sys} / 0.0275$

2. $t = T_{sys}{}^2 \times 1322.3$

   $T_{sys} = 100K \Rightarrow 3673h$

   $T_{sys} = 10K \Rightarrow 36.73h$

Homework (Assigned February 2, due February 14): Using a 100-meter radio telescope, calculating the beam in radians as the diffraction limit ($1.22\lambda/d$ where λ is the wavelength and d the telescope diameter), what T_{sys} is needed to detect Jupiter to 10% @ 10 cm wavelength in 5 hours? Ignore the CMB.

At what angular resolution does Jupiter match the CMB? If I include the CMB, can I make the measurement with this telescope?