$$I = \int\int\int \rho \left(x^2 + y^2\right) \mathrm{dV}$$

$$= \int\int\int \rho r^2 \left(x^2 + y^2\right) dr d\Omega$$

$$= \int\int \rho r^2 \left(r^2 \sin^2\theta\right) dr \left(\sin\theta\right) d\theta$$

$$= 2\pi \int\int \rho r^4 \left(1 - \cos^2\theta\right) d\left(\cos\theta\right) dr$$

$$= \frac{8\pi}{3} \int \rho r^4 dr = \frac{8\pi}{15} \rho R^5$$

$$= \frac{2}{5} M R^2 .$$

4.2.7 Total angular momentum of the Jupiter-Sun System

The total angular momentum is the sum of the orbital (rotational) angular momentum plus the spin momenta of Jupiter and the Sun.

Jupiter is the largest planet in the solar system with mass $M_J = 0.95\times10^{-3} M_\odot$ and radius $R_J = 7.14\times10^9$ cm. The semi-major axis of Jupiter's orbit is a = 5.2 AU, its orbital eccentricity $\varepsilon = 0.048$ and its orbital period is $P = 11.86$ yr. Its orbital angular momentum is given by (see 4-30)

$$L = \mu G\{(M_\odot + M_J)\, a\, (1-\varepsilon^2)\}^{1/2}$$

$$= 1.94 \times 10^{50} \text{ g cm}^2 \text{ s}^{-1} .$$

The distance of the Sun from the center of mass is

$$a_\odot = \frac{aM_J}{M_J + M_\odot} = 7.4 \times 10^{10} \text{ cm}$$

(slightly larger than the Sun's radius). Assuming that the Sun moves in a circular orbit, its angular momentum is

$$L_\odot = M_\odot \text{v}_\odot a_\odot = \frac{M_\odot 2\pi a_\odot^2}{P}$$

$$= 1.84 \times 10^{47} \text{ g cm}^2 \text{ s}^{-1} \ .$$

(Note: $L_\odot$ is not the solar luminosity here!)

The distance from Jupiter to the center of mass is $a_J = a - a_\odot \sim a$. So its angular momentum is

$$L_J = M_J \text{v}_J a_J = M_J = \frac{M_\odot 2\pi a^2}{P}$$
$$= 1.94 \times 10^{50} \text{ cm}^2 \text{ s}^{-1} \ .$$

So $L = L_\odot + L_J$ (as it must).

The spin angular momentum of the Sun is

$$L_\odot^s = I_\odot \omega_\odot = I_\odot \frac{2\pi}{P_\odot}$$

$$= \frac{4\pi}{5} \frac{M_\odot R_\odot^2}{P_\odot} \ .$$

$P_\odot = 26$ days, $R_\odot = 6.696 \times 10^5$ km, $M_\odot = 1.99 \times 10^{33}$ g.

$$L_{\odot}^{s} = 9.9 \times 10^{48}\ \mathrm{g\ cm^2\ s^{-1}}\ .$$

Similarly, using $P_J = 10$ hours

$$L_J^{\ s} = 6.7 \times 10^{45}\ \mathrm{g\ cm^2\ s^{-1}}\ .$$

Angular momentum of the Sun-Jupiter system is dominated by the orbital motion but if the Sun were a rapidly rotating star, its spin could dominate.

4.3 Binary stars

(See Ostlie and Carroll, Chapter 7)

More than half the stars are multiple stars, most of which are binary pairs. For solar-type stars, the observed ratios of single: double: triple: quadrupole systems is 45:46:8:1. There are several classes of binaries:

visual binaries: both can be detected orbiting in ellipses about one another (Sirius is a famous example - Sirius A is a main sequence star of spectral type A1, Sirius B is a white dwarf of spectra type A5. The period is 49.9 years.

Sirius was first discovered as an **astrometric binary**.

Astrometric binaries are binaries in which only one star is observed but its motion is oscillatory, indicating the perturbing presence of a dim companion.

Spectroscopic binaries are visually unresolved but periodic oscillations occur in their spectrum. If only one stellar spectrum is observed, the binary is *single-lined*; if both are observed, the binary is double-lined.

Eclipsing binaries occur when the two stars eclipse one another, producing periodic changes in apparent brightness.

Periods of binary stars vary from a few hours to hundreds of years. From data on the periods we can use the law of gravitation to infer masses. Consider a double-lined spectroscopic binary. The spectra of the two stars are superimposed. We can use Doppler shifts to measure the radial velocities of each star, even though they may be too close for their orbits to be distinguished.

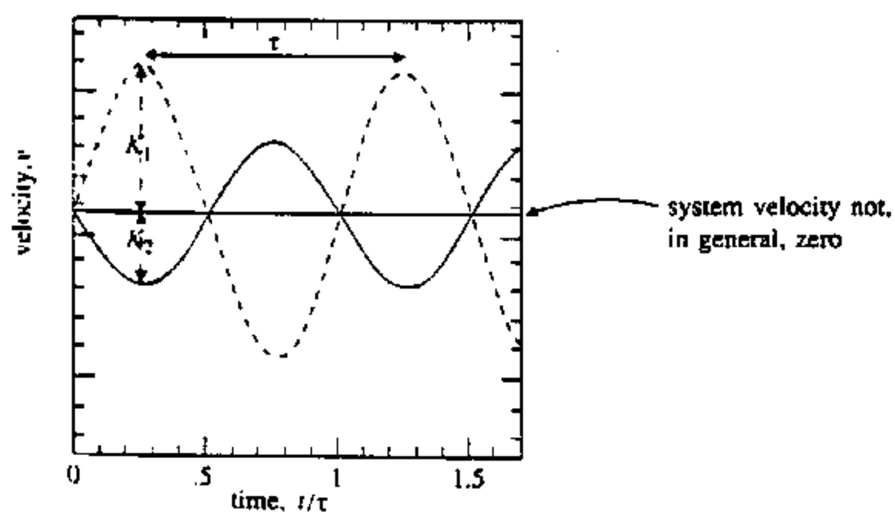


Fig. 4-12

The line joining the stars is rotating with angular velocity Ω and $\mathbf{K}_1$ and

$\mathbf{K}_2$ are the inferred radial velocities.

The figure shows the individual radial velocities. From it we obtain the peak velocities of each star and the binary period τ. If the shape is accurately sinusoidal, the orbits are circular with $\varepsilon = 0$. The distances r_1 and r_2 from the center of mass are constant for circular orbits. The center of mass is the center of the orbits of both stars and of their relative motion.

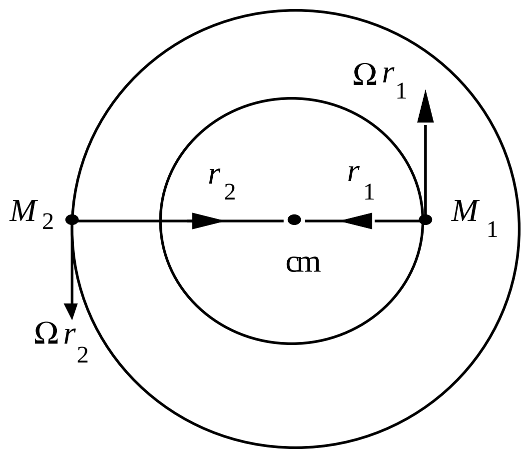


Fig. 4-13

$$r_1 = \frac{M_2}{M_1 + M_2} r \qquad r_2 = \frac{M_1}{M_1 + M_2} r \quad .$$

The distance of M_1 from M_2 is $r_1 + r_2$.

The period and the separation are related by Kepler's Third Law

$$\tau^2 = \frac{4\pi^2 r^3}{GM} = \frac{4\pi^2 (r_1 + r_2)^3}{G(M_1 + M_2)}$$

and the speeds of the stars are

$$v_1 = \Omega r_1 , \quad v_2 = \Omega r_2$$

where Ω is the angular velocity

$$\Omega = \frac{2\pi}{\tau} .$$

So $r = r_1 + r_2 = (v_1 + v_2)/\Omega$

$$\frac{M_1}{M_2} = \frac{r_2}{r_1} = \frac{v_2}{v_1} \text{, independent of } i$$

$$(M_1 + M_2) = \frac{\Omega^2 r^3}{G} .$$

The peak velocities equal v_1 and v_2 only if the orbital plane is parallel to the line of sight. If i is the inclination angle between the line of sight and the normal to the orbital plane = the angle between the plane of the sky (defined as perpendicular to the line of sight) and the plane of the orbit

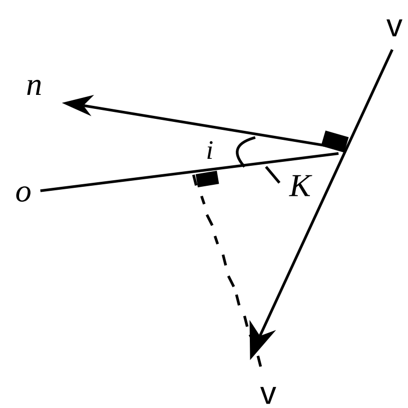


Fig. 4-14

$K_1 = \mathrm{v}_1 \sin i \;= \Omega r_1 \sin i = \dfrac{2\pi}{\tau} r_1 \sin i$

$K_2 = \mathrm{v}_2 \sin i \;= \Omega r_2 \sin i = \dfrac{2\pi}{\tau} r_2 \sin i$

If $i = 0$, we observe zero velocities (no information). $i = 90°$ is edge on $K_1 = \mathrm{v}_1$, $K_2 = \mathrm{v}_2$. The mass ratio is in any case independent of i.

$$\frac{M_1}{M_2} = \frac{K_2}{K_1} = \frac{\mathrm{v}_2}{\mathrm{v}_1} .$$

The separation $r = r_1 + r_2 = \dfrac{\tau\left(K_1 + K_2\right)}{2\pi \sin i}$

The total mass we obtain from

$$M = \frac{\Omega^2 r^3}{G} = \frac{4\pi^2}{\tau^2} \frac{r^3}{G}$$

$$M = M_1 + M_2 = \frac{\tau}{2\pi G} \frac{\left(K_1 + K_2\right)^3}{\sin^3 i} .$$

Hence we may write for the case when only K_2 can be measured,

$$M_1 + M_2 = \frac{\tau}{2\pi G} \frac{1}{\sin^3 i} \frac{(M_1 + M_2)^3}{M_1^3} K_2^3$$

$$M_1 + M_2 = \frac{\tau}{2\pi G} \frac{1}{\sin^3 i} \left(1 + \frac{M_2}{M_1}\right)^3 K_2^3 \; .$$

If we write this equation in the form

$$\frac{\tau}{2\pi G} K_2^3 = \frac{(M_1 \sin i)^3}{(M_1 + M_2)^2} = f(M)$$

we define a function $f(M)$ called the mass function. The left-hand side depends only on observable properties and is useful for statistical studies (Ostlie and Caroll p. 211).

We may write for the separate masses,

$$M_1 \sin^3 i = \frac{\tau}{2\pi G} (K_1 + K_2)^2 K_2$$

$$M_2 \sin^3 i = \frac{\tau}{2\pi G} (K_1 + K_2)^2 K_1$$

but in general we do not know i.

For eclipsing binaries, each star successively eclipses the other. To see them, i must be near 90°, assuming that the stellar radii are much less than the stellar separation. Masses are insensitive to i for i near 90° since $\sin i \sim 1$.

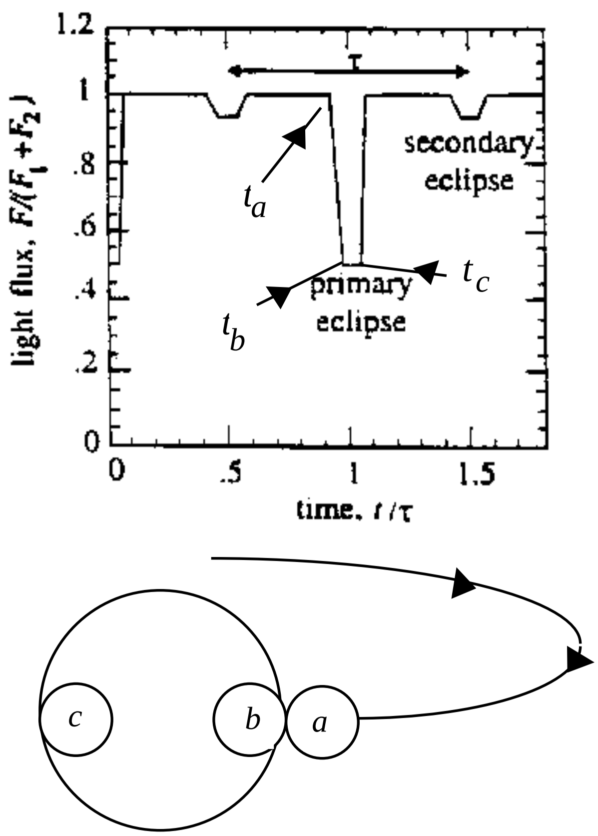


Fig. 4-15

From the duration of the eclipses we can infer the radii of the stars. Assume i = 90°. Let t_a be the time of first contact of the primary eclipse and let t_b be the time of minimum light. During eclipses stars are moving nearly perpendicular to the line of sight to the observer in opposite directions. Thus, if v_s and v_l are the velocities of the small and large stars respectively, their relative velocity is $v_s + v_l$ The radius of the smaller star is given by

$$r_s = \frac{1}{2}(\mathrm{v}_s + \mathrm{v}_l)(t_b - t_a)$$

and the radius of the larger star by

$$r_l = \frac{1}{2}(\mathrm{v}_s + \mathrm{v}_l)(t_c - t_b)$$

where $t_c - t_b$ is the duration of minimum light (see Ostlie and Carroll §7.1)

4.4 Extrasolar planets

It is primarily by velocity measurements of the parent star that extra solar planets have been detected and we do not know the planet velocity. Let M_s be the stellar mass and M_p the planetary mass. Suppose τ is the measured period and K_s the measured stellar velocity. The mass function $f(M)$ is

$$\frac{\left(M_p \sin i\right)^3}{\left(M_p + M_s\right)^2} = \frac{\tau K_s^3}{2\pi G}$$

If we assume $M_s >> M_p$,

$$\left(M_p \sin i\right)^3 = \frac{\tau K_s^3}{2\pi G} M_s^2 \ .$$

Observations of the star Peg A—a G star like the Sun—have revealed the existence of a companion and τ and K_s have been measured. From them we get

$$M_p \sin i = (1.36 \times 10^{20}\, M_s^{\,2})^{1/3}$$

The mass of Peg A is 0.95 $M_\odot \sim 2 \times 10^{30}$ kg so if $\sin i = 1$

$$M_p = 8 \times 10^{26}\text{ kg} .$$

It is customary to express the mass of extra solar planets in Jupiter masses

$$M_p = 0.45\, M_J .$$

The orbital radius is obtained from

$$a = K_s \tau / 2\pi$$

It equals 3000 km. From

$$\frac{a_p}{a} \sim \frac{M_s}{M_p} \quad ,$$

we obtain for the distance of the planet from the star a value of 0.05 AU. The value of M_p and a_p depend on the assumption that $i = 90°$. M_p could be underestimated and a_p overestimated.

4.5 Supernovae in binary systems

Supernovae are exploding stars. Before explosion many occur as binary systems and are caused by mass flow from a companion star. What happens to the binary system when the explosion occurs and the mass of one star is reduced, possibly to zero?

Before, there are two stars in circular orbit

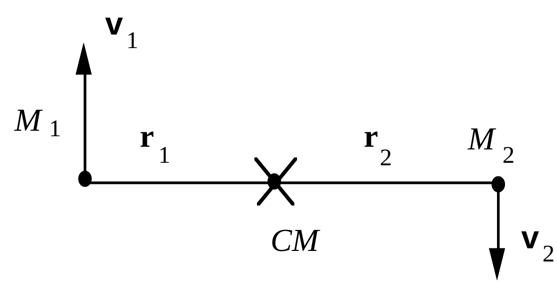


Fig. 4-16

Assume center of mass is at rest, take it as origin

$$M_1 \mathbf{r}_1 + M_2 \mathbf{r}_2 = 0$$
$$M_1 \mathbf{v}_1 + M_2 \mathbf{v}_2 = 0$$

$M_1 > M_2$ explodes, leaving new mass $M_1' = M_1 - \Delta M$. Remaining binary is not at rest. In a spherical explosion, linear momentum carried away is zero. If $\mathbf{v}_c$ is the new *CM* velocity, momentum conservation yields

$$M_1' \mathbf{v}_1 + M_2 \mathbf{v}_2 = (M_1' + M_2)\mathbf{v}_{cm}$$

($\mathbf{v}_1$ and $\mathbf{v}_2$ are the same immediately before and after the explosion and total momentum is the same as the total mass moving with the cm velocity.) Using $\mathbf{v}_1 = -M_2 \mathbf{v}_2 / M_1$ and $M_1' = M_1 - \Delta M$,

$$(M_1 - \Delta M)\left(\frac{-\mathrm{M}_2}{\mathrm{M}_1}\ \mathbf{v}_2\right) + M_2\,\mathbf{v}_2$$

$$= (M_1 - \Delta M + \ M_2\,)\mathbf{v}_{cm}$$

giving

$$\mathbf{v}_{cm} = \frac{\Delta M\; M_2}{M_1\left(M_1 + \ M_2\ -\ \Delta M\ \right)}\ \mathbf{v}_2\ .$$

If $\Delta M\ = M_1$, $\mathbf{v}_c = \mathbf{v}_2$ (as it must).

Typical values are $M_1 = 10\ M_\odot$, $M_2\ = 5\ M_\odot$, $\Delta M = 8.5\ M_\odot$. Then $M_1' =$ 1.5 $M_\odot$ (appropriate for a neutron star).

$$\mathrm{v}_{cm}\ =\ \frac{8.5 \text{ x } 5}{10 \text{ x } 5.5}\mathrm{v}_2\ =\ 0.77\,\mathbf{v}_2\ .$$

For close binaries, $\mathbf{v}_2$ may be several hundred km s^{-1} so system really moves.

To determine whether or not the binary remains bound, calculate the binding energy, that is, the internal energy without the center of mass energy.

The total *internal* energy of the system immediately after the explosion is

$$E' = \left(\frac{1}{2} M'_1\, \mathrm{v}_1^2 + \frac{1}{2} M_2\, \mathrm{v}_2^2\right) - \frac{1}{2}\left(M'_1 + M_2\right)\mathrm{v}_{cm}^2 - \frac{GM'_1 M_2}{r}$$

Total kinetic energy — Energy of CM motion — Gravitational Energy

Now $\dfrac{G(M_1 + M_2)}{r} = \left(\mathrm{v}_1 - \mathrm{v}_2\right)^2$ for circular orbits.

We obtain, writing everything in terms of v_2

$$E' = \frac{1}{2}\left(M_1 - \Delta M\right)\left(\frac{M_2 \mathrm{v}_2}{M_1}\right)^2 + \frac{1}{2} M_2 \mathrm{v}_2{}^2$$

$$- \frac{1}{2}\left(M_1 + M_2 - \Delta M\right)\left\{\frac{\Delta M\, M_2 \mathrm{v}_2}{M_1\left(M_1 + M_2 - \Delta M\right)}\right\}^2$$

$$- \frac{\left(M_1 - \Delta M\right) M_2}{M_1 + M_2}\left[\left\{1 + \frac{M_2}{M_1}\right\}\mathrm{v}_2\right]^2$$

which (believe it or not!) simplifies

$$E' = \frac{1}{2}\left(M_2 \mathrm{v}_2{}^2\right)\frac{\left(M_1 - \Delta M\right)\left(M_1 + M_2\right)}{\mathrm{M}_1^2\left(M_1 + M_2 - \Delta M\right)}\left(M_1 + M_2 - 2\Delta M\right).$$

All terms are positive except the last factor.

Thus for E' to be positive (no binding) , mass ejected $\Delta M > 1/2\ (M_1 + M_2)$.

In the numerical example on p 4.52

$$8.5 > \frac{1}{2}(10 + 5)$$

and the neutron star departs at high velocity.

Pulsars (rotating neutron stars) often have high velocity as they leave the galactic plane.

The result can be obtained more readily using the *CM* system in which the total energy is

$$E_{tot} = \frac{1}{2} M \mathrm{v}^2_{\mathrm{cm}} + \mu \left(\frac{1}{2}\mathrm{v}^2 - GM/r\right)$$

where v is the relative velocity. Before the explosion for the initial circular orbit

$$\mathrm{v}^2 = \frac{GM}{r}$$

and

$$\frac{E}{\mu} = \frac{1}{2}\mathrm{v}^2 - \frac{GM}{r} = -\frac{GM}{2r} .$$

(see page 4-29) where E is internal energy.

After explosion, v is unchanged—μ, M and E change as M changes to M'. Internal energy is changed from $\frac{1}{2}\mathrm{v}^2 - \frac{GM}{r}$ to

$$\frac{E}{\mu'} = \left(\frac{1}{2}\mathrm{v}^2 - \frac{GM'}{r}\right)$$

$$= \frac{GM}{2r} - \frac{GM'}{r} \quad .$$

So internal energy > 0 if $M' < M/2$ and ejected mass $\Delta M > M/2$.

4.6 Tides

When two bodies are in orbit around each other, the otherwise spherically symmetric gravitational field is distorted by the gravitational attraction of the other body. The force field can be characterized by equipotentials which are like contours of height on a map; the force is zero tangent to the equipotential surface and is normal everywhere to the surface.

4.6.1 Weak tides

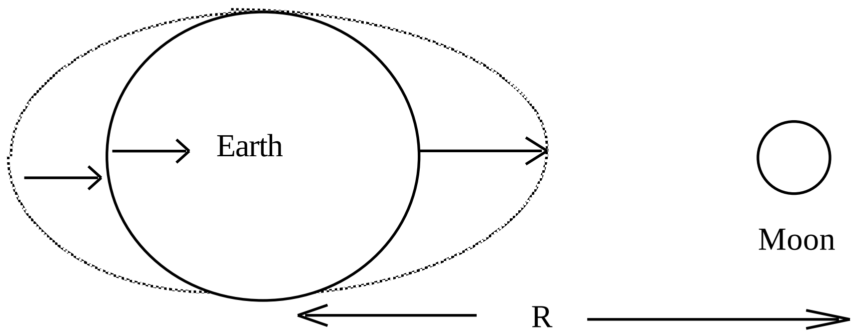


Fig. 4-17

For the Earth-Moon system, the Moon pulls the near surface most strongly, the center of the Earth less strongly and the far surface least strongly. The differential force gives rise to ocean tides.

The ocean surface adjusts to become an equipotential. The potential is formed by the gravitational attraction and by the centrifugal force that arises because the Earth-Moon system is orbiting about the center of mass.

Assume masses are concentrated at the centers of the Earth and Moon. The gravitational potential at a point **r** for a unit test mass is

$$V(\mathbf{r}) = \frac{-GM_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{GM_2}{|\mathbf{r} - \mathbf{r}_2|}$$

to which must be added the centrifugal potential arising because of the rotating frame. It is (from 4-19)

$$-\ \frac{1}{2}\ r^2\,\dot{\theta}^2\ =\ -\ \omega^2 r^2/2$$

where ω is the angular velocity

$$\omega^2 = \frac{G(M_1+M_2)}{R^3}$$

R being the Earth-Moon distance.

(Remember $\frac{1}{2}\mu\dot{r}^2 = E - \frac{1}{2}\mu r^2\ \dot{\theta}^2 + \frac{G\mu M}{r} = E - \frac{1}{2}\mu r^2\ \omega^2 + \frac{G\mu M}{r}$.)

We first evaluate V(**r**) at a point r.

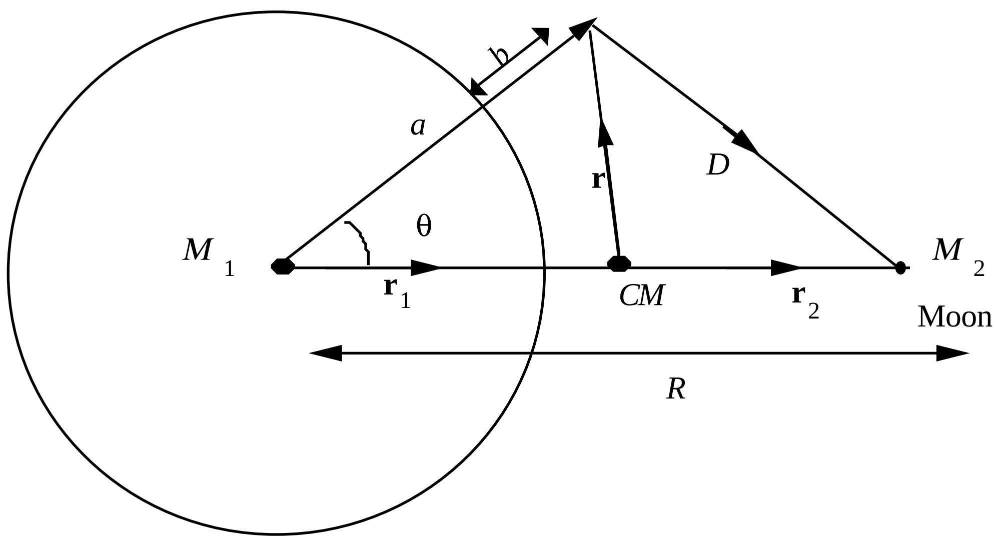


Fig. 4-17

Moon, Earth are small compared to Earth-Moon distance R so we write

$$\frac{-GM_2}{|\boldsymbol{r}-\boldsymbol{r}_2|} = \frac{-GM_2}{D} = \frac{-GM_2}{\left(R^2 + a^2 - 2aR\cos\theta\right)^{1/2}}$$

$$= -\frac{GM_2}{R}\left(1 + \frac{a^2}{R^2} - \frac{2a}{R}\cos\theta\right)^{-1/2}.$$

Binomial expansion $\frac{1}{(1+x)^{1/2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + 0\left(x^3\right).$

Thus $\frac{GM}{D} \sim -\frac{GM_2}{R}\left(1 - \frac{1}{2}\frac{a^2}{R^2} + \frac{a}{R}\cos\theta + \frac{3a^2}{2R^2}\cos\theta^2 + O\left(\frac{a}{R}\right)^3\right)$

$$= -\frac{GM_2}{R}\left\{1 + \frac{a}{R}P_1\left(\cos\theta\right) + \frac{a^2}{R^2}P_2\left(\cos\theta\right) + \ldots\right\}.$$

(Alternatively use expansion

$$\frac{1}{|\boldsymbol{R} - \boldsymbol{a}|} = \frac{1}{R}\sum_n \left(\frac{a}{R}\right)^n P_n(\cos\theta) \qquad R > a$$

where $P_n(\cos\theta)$ are Legendre polynomials.) The term in the potential $\frac{a}{R}\cos\theta$ is linear in z, where z is the direction from M_1 to M_2 so its gradient describes a constant force—GM_2/R^2 which must be canceled by the centrifugal potential.. The centrifugal potential can be written, using $\mathbf{r}_1 = \frac{M_2}{M_1 + M_2}\mathbf{R}$,

$$- \frac{1}{2}\omega^2 r^2 = \frac{1}{2}\omega^2 \left|\mathbf{r}_1 - \mathbf{a}\right|^2 = \frac{1}{2}\omega^2 \left(r_1^2 + a^2 - 2\boldsymbol{r}_1 a \cos\theta\right)$$

$$= \frac{1}{2}\omega^2 \left[\left(\frac{M_2}{M_1 + M_2}\right)R^2 + a^2 - \frac{2M_2}{M_1 + M_2} R\, a \cos\theta\right].$$

So adding this we have for the total potential $\Phi(\mathbf{r})$

$$\Phi(\boldsymbol{r}) = -\frac{GM_1}{a} - \frac{GM_2}{R}\left[1 + \frac{a}{R} \cos\theta + 1/2\,(3\cos^2\theta - 1)\,\frac{a^2}{R^2}\right]$$

$$- \frac{1}{2}\frac{G\left(M_1 + M_2\right)}{R^3}\left[\left(\frac{M_2}{M_1 + M_2}\right)^2 R^2 + a^2 - 2\left(\frac{M_2}{M_1 + M_2}\right)R\,a\cos\theta\right].$$

The term in $\frac{a}{R}\cos\theta$ is indeed canceled out by the centrifugal force that keeps the body in a circular orbit. The gradient of terms that do not depend on a or θ is zero, so they may be omitted and we have for the local tidal potential

$$\Phi(\boldsymbol{r}) = -\frac{GM_1}{a} - \frac{1}{2}\frac{Ga^2}{R^3}\left(3M_2\cos^2\theta + M_1\right).$$

Expand a in terms of its height above the mean sea level $a = R_\oplus + h$. Then

$$a^2 \simeq R_\oplus^{\,2}\left(1 + \frac{2h}{R_\oplus}\right),$$

$$a^{-1} \simeq R_\oplus\left(1 - \frac{h}{R_\oplus}\right)$$

and

$$\Phi(h,\theta) = -\frac{GM_1}{R_\oplus}\left(1 - \frac{h}{R_\oplus}\right) - \frac{1}{2}\frac{GR_\oplus^2}{R^3}\left(1 + 2\frac{h}{R_\oplus}\right)\left(3M_2\cos^2\theta + M_1\right)$$

$$\sim \frac{GM_1}{R_\oplus^{\,2}}\left[h - \frac{3}{2}\left(\frac{M_2}{M_1}\right)\left(\frac{R_\oplus}{R}\right)^3 R_\oplus\cos^2\theta\right]$$

ignoring constant terms.

$\frac{GM_1}{R_\oplus^{\,2}}$ = g = acceleration due to gravity at the surface of the Earth.

The surface will adjust to be an equipotential (the tangential force vanishes) so

$$\Phi(h, \theta) = \text{constant}$$

$$h = \frac{3}{2} \left(\frac{M_2}{M_1}\right) \left(\frac{R_\oplus}{R}\right)^3 R_\oplus \cos^2 \theta$$

$$+ \text{ constant} \quad .$$

The height of the tides is the difference between high and low values of *h*. Since $\cos^2\theta$ varies between 1 and 0, we get for the height of the tides with

$$\frac{M_2}{M_1} = \frac{1}{81} \, , R_\oplus = 6000 \text{ km} = \text{radius of Earth}$$

$$R = 380{,}000 \text{ km} = \text{Earth} - \text{Moon distance}$$

$$\text{Then } h = 54 \text{ cm} \, .$$

The same calculation with the Sun in place of the Moon yields

$$h = \frac{3}{2} \text{ x } (332000) \text{ x} \left(\frac{6400}{1.5 \text{ x } 10^8}\right)^3 \text{ x } 6400 \text{ km} = 25 \text{ cm}$$

(presumably by chance, they are of the same order). The tidal effects combine vectorially. When the Moon is at conjunction or opposition, the two forces add to cause high tides.

4.6.2 Tidal friction

The continents are pulled through the ocean bulges and the tidal bulge is dragged ahead by the spinning Earth. There is a loss of energy by friction and the spin of the Earth is slowed. The day is getting longer. (There is evidence from growth scales in fossil corals that there were 400 days in a year about 100 million years ago). Angular momentum is conserved so the Moon increases its angular momentum. It can do so because the non-symmetric bulge creates a gravitational torque back on the Moon. Increasing the angular momentum means the Moon must move outward $\left(r_o = L^2/Gm\mu^2, a = \frac{r_o}{(1-\varepsilon^2)}\right)$ and so the month is getting longer. The lowest energy state of the Moon-Earth system is one in which the Earth and Moon present the same face—in which case the tidal distortion will have reached its equilibrium shape that involves no relative motion of any material. The Earth and Moon will be *tidally locked* and there will be no drag. The tidal bulge will point directly at the Moon and the Earth and the Moon will corotate.

Because the Moon is not exactly spherical, partial locking has already occurred, in that the Moon rotates with the Earth so that it shows the same face all the time. The Moon is in synchronous rotation such that the orbital period of the Moon around the Earth equals the rotation or spin period of the Moon. The ultimate equilibrium caused by tidal friction is that in which the spin velocity of the Earth equals the angular velocity of the Moon in orbit around the Earth (or the angular velocity of the Earth about the moon) so that 1 month equals 1 day.

To calculate when that equilibrium will be reached, use the conservation of angular momentum. The angular momentum of the Earth-Moon system is the sum of the angular momentum of the spinning Earth and Moon and the angular momentum of the Moon's orbit around the Earth.

The angular momentum of the Earth may be written $I_1\omega_1$ where I_1 is the moment of inertia and ω_1 is the spin angular velocity. From p. 4-40, $I_1 = \frac{2}{5}MR^2$. The total angular momentum is

$$I_1\omega_1 + I_2\omega_2 + \frac{M_1 M_2}{M}\,\omega_2 R_o^{\,2}$$

(see 4.2.7)

(note the Moon spin and orbital angular velocities are equal) and eventually is

$$I_1\omega_f + I_2\omega_f + \frac{M_1 M_2}{M}\,\omega_f R_f^{\,2}$$

where $\omega_f = 2\pi/$(ultimate day or month), R_f the ultimate Earth-Moon distance. (1) refers to the Earth and (2) to the Moon and R_o = 380,000 km is the present Earth-Moon distance.

Kepler's law gives R_f

$$\frac{R_f}{R} = \left(\frac{\omega_2}{\omega_f}\right)^{2/3}.$$

Now $M_1 \sim 81.3\, M_2$ so $\frac{M_1 M_2}{M} \sim M_2$.

Also $I_2\omega_2$ is small compared to $I_1\omega_1$ and as we can check once we have the answer, both spins are negligible in the final state. Then the initial angular momentum can be approximated by

$$\frac{2}{5} M_1 R_1{}^2 \omega_1 + M_2 \omega_2 R_o{}^2$$

and the final angular momentum by

$$M_2 \omega_f R_f{}^2 = M_2 \omega_2 R_o{}^2 \left(\frac{\omega_2}{\omega_f}\right)^{1/3}$$

Hence

$$\frac{\omega_2}{\omega_f} = \left\{\frac{\frac{2}{5} M_1 R_1{}^2 \omega_1 + M_2 \omega_2 R_o{}^2}{M_2 \omega_2 R_o{}^2}\right\}^3$$

$$= \left\{1 + \frac{2}{5} \left(\frac{M_1}{M_2}\right) \left(\frac{R_1}{R_o}\right)^2 \left(\frac{\omega_1}{\omega_2}\right)\right\}^3$$

$$= \left\{1 + \frac{2}{5} \times 81.3 \left(\frac{6400}{380,000}\right)^2 \times 28\right\}^3$$

$$= 1.99$$

The final length of the day and month will be 28 x 1.99 = 54 days.

The current lengthening of the day is about 0.2 days in 10^9 years so it will take more than 10^{10} years to reach equilibrium. (We will have been engulfed by the Sun in its evolution by then).

The same tidal forces bring binary stars into corotation, tidally locked to each other.

4.7 Roche stability limits for satellites

Objects can be torn apart by tidal forces. We give an approximate description. Tidal potential at the point (r,θ) is

$$\Phi(r,\theta) = \frac{-GM_1}{r} - \frac{1}{2}\frac{Gr^2}{R^3}\left(3M_2\cos^2\theta + M_1\right) + \text{constant}$$

Suppose M_1 is the mass of a small satellite orbiting a large parent star or planet of mass M_2. $M_1 \ll M_2$.

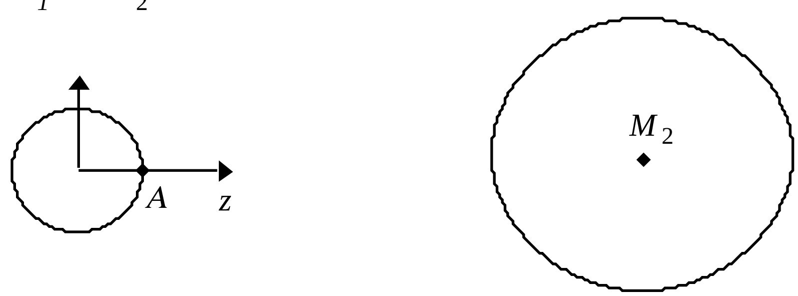


Fig. 4-18

Calculate the force at point *A*. If it points towards the center of the satellite, it is a restoring force. If it points away towards M_2, the satellite is torn apart.

Gravitational acceleration along the *z* axis at *A* is given by

$$g_z = -\nabla_z \Phi = -\frac{d}{dz}\left[-\frac{GM_1}{|z|} - \frac{1}{2}\frac{Gz^2}{R^3}\left(3M_2\right)\right]$$

$$= -\frac{GM_1}{|z|^3}z + z\frac{3GM_2}{R^3}$$

$$= Gz\left(-\frac{M_1}{|z|^3} + \frac{3M_2}{R^3}\right).$$

The condition for *Roche stability* is

$$\frac{M_1}{r^3} > \frac{3M_2}{R^3}$$

Putting $|z|=r$, for a satellite of mean density ρ we obtain

$$\rho > \frac{9}{4\pi}\frac{M_2}{R^3}$$

or no satellite of density ρ is stable inside the *Roche radius* R_{crit}. The critical radius is

$$R_{crit} = \left(\frac{9M_2}{4\pi\rho}\right)^{1/3} \quad .$$

If the parent (planet) and satellite have same density

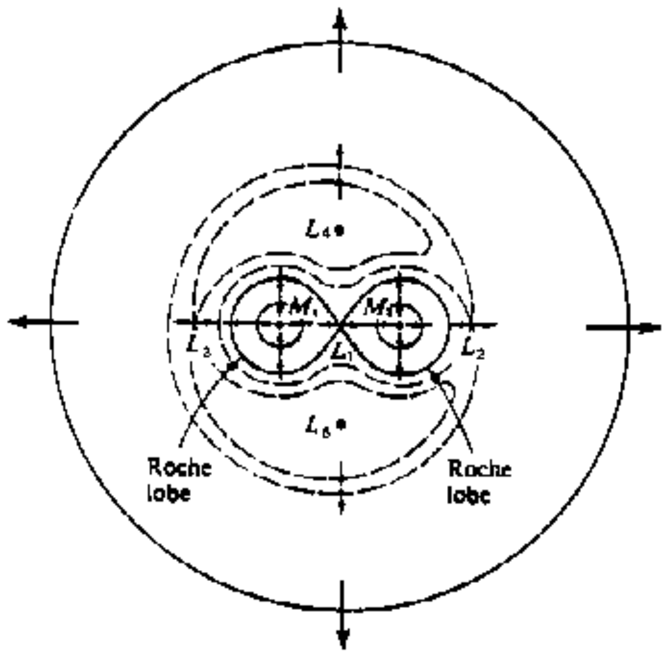
$$R_{\mathrm{crit}} = 3^{1/3} R_2 = 1.44\, R_2$$

where R_2 is radius of the parent. (Roche calculated 2.44 R_2 using a model of the self-gravity of the satellite).

This is essentially the physics of the rings of Saturn (and other planets). Material within the Roche limit cannot form bodies such as moons because of the disruptive effect of tidal forces.

4.8 Roche lobes

Consider a binary stellar system in a circular orbit. The intersections of the equipotential surfaces with the plane of the orbit are shown in the Figure. (See Ostlie and Caroll p. 687).



The Roche equipotential surfaces plotted in the equatorial plane for two point mass with a mass ratio equal to 2/3. The short arrows indicate the direction of the effective gravitational field in the frame of reference which corotates with the orbital motion. The effective gravity vanishes at the five Lagrangian points L_1, L_2, L_3, L_4, L_5. The first three, L_1, L_2, L_3, lie along the line joining the two mass points; the last two, L_4, L_5, form equilateral triangles with the two mass points, M_1 and M_2. The sideways "figure 8" which passes through the L_1 point contains the two Roche lobes.

Fig. 4-19

There are five stationary points, called Lagrangian points, where the force vanishes. Close to each star, the equipotentials are dominated by the gravitational attraction and the equipotentials are circles centered at the stars (taken to be point sources). Far from the stars, the equipotentials are dominated by the outwardly directed centrifugal force. There the equipotentials intersect the equatorial plane in circles enclosing both stars. The two kinds of equipotentials are separated by *Roche* *lobes* around each star indicated by the figure of eight. The Roche lobes intersect at a saddle point, forming in a pitcher-like shape.

Roche lobes can be used to further classify close binaries. If both stars are smaller than their Roche lobes, the system is a *detached* binary. If one fills its Roche lobe, the system is a *semi-detached* binary and matter will flow through the contact point. If both stars fill their Roche lobes they are *contact binaries* and they have a common envelope.

The Roche lobe is the maximum possible size of the star. If a star of mass M_1 becomes larger than its Roche lobe, it overflows and dumps mass through the saddle-point on to the companion star. A common scenario is the case where M_1 is initially much larger than M_2 (possibly also losing mass to infinity in a stellar wind). Mass flows from M_1 to M_2. Eventually M_1 becomes a white dwarf and cools. M_2 has gained mass and so it evolves faster and overflows back on to M_1. This process manifests itself in an X-ray source. As the white dwarf accumulates mass, it may be forced into a gravitational collapse to a neutron star in a supernova explosion.

4.8.1 Effect of mass transfer on binary orbits

Suppose M_1 is filling its Roche lobe and dumping mass on to M_2. Mass and angular momentum are conserved but not energy. The mass is heated and dissipates energy in radiation.

M_2 is gaining mass so $\dot{M}_2 = -\dot{M}_1 > 0$. The angular momentum for a circular orbit

$$L = \mu R^2 \omega = \mu R^2 \left(\frac{GM}{R^3}\right)^{1/2}$$

$$= \frac{M_1 M_2}{M^{1/2}} R^{1/2} G^{1/2} = \left(M_1 M_2 R^{1/2}\right)\left(G^{1/2} M^{-1/2}\right)$$

$$0 = \frac{dL}{dt} = \left(G^{1/2} M^{-1/2}\right)\left(\dot{M}_1 M_2 R^{1/2} + M_1 \dot{M}_2 R^{1/2}\right)$$

$$+ \left(G^{1/2} M^{-1/2}\right)\frac{1}{2} M_1 M_2 \dot{R} \; R^{-1/2} .$$

Solving for $\dot{R}$ and eliminating $\dot{M}_1$ in favor of $\dot{M}_2$, we obtain

$$\dot{R} = 2R\left(\frac{M_2 - M_1}{M_1 M_2}\right)\dot{M}_2 .$$

If the lighter star M_1 is losing mass $\dot{R} > 0$ and stars draw apart. Often this terminates the mass flow since it puts M_1 deeper into its Roche lobe. Alternatively the mass transfer proceeds slowly on the stellar evolutionary time scale that it takes M_1 to fill its increasingly large Roche lobe.

If the heavier star is losing mass $\dot{R}$ is negative. The stars get nearer which increases mass flow leading to a catastrophic instability. In practice friction leads to a merger of the two stars.

4.9 The Virial Theorem

Here I prove a useful theorem, the *virial theorem.*

Introduce

$$I = \frac{1}{2}\sum_{i=1}^{N} m_i r_i^2$$

(similar to moment of inertia but about a point).

The kinetic energy of N interacting particles of masses m_i and velocities v_i is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \mathsf{v}_i^2$$

The gravitational potential energy is

$$\mathsf{V} = -\sum \sum_j \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Newton's law

$$m_i \dot{\mathsf{v}}_i = -\nabla_i \mathsf{V}$$

Differentiate *I* with respect to time twice

$$\dot{I} \;=\; \sum_{i=1}^{N} m_i \mathrm{v}_i \cdot r_i$$

$$\ddot{I} = \sum_{i=1}^{N} m_i \dot{\mathrm{v}}_i \cdot r_i + \sum_{i=1}^{N} m_i \mathrm{v}_i \cdot \mathrm{v}_i$$

$$= -\sum_i r_i \cdot \nabla_i \mathrm{V} + 2\mathrm{T}\,.$$

This is the time-dependent virial theorem. To evaluate

$$\sum_i \mathbf{r}_i \cdot \nabla_i \mathrm{V}$$

consider the scaled potential $\mathrm{V}(\lambda \mathbf{r}_i)$ obtained by replacing all position vectors by $\lambda \mathbf{r}_i = \boldsymbol{\rho}_i$, say. Then $\mathrm{d}\boldsymbol{\rho}_i / d\lambda = \mathbf{r}_i$ and

$$\frac{d}{d\lambda}\,\mathrm{V}\left(\lambda \mathbf{r}_i\right) \;=\; \frac{d}{d\lambda}\,\mathrm{V}\left(\boldsymbol{\rho}_i\right) \;=\; \frac{\mathrm{dV}\left(\boldsymbol{\rho}_i\right)}{\mathrm{d}\left(\boldsymbol{\rho}_i\right)}\,\frac{d\boldsymbol{\rho}_i}{d\lambda} \;= \nabla_i \mathrm{V} \cdot \mathbf{r}_i\;\;.$$

Thus

$$\frac{d}{d\lambda} V(\lambda \mathbf{r}) = \sum_i \mathbf{r}_i \cdot \nabla_i V \ .$$

For a gravitational potential

$$V(\lambda \mathbf{r}) = \frac{1}{\lambda} V(\mathbf{r})$$

so

$$\sum_i r_i \cdot \nabla_i V = -\frac{1}{\lambda^2} V(\mathbf{r}) \ .$$

Put $\lambda = 1$. Then

$$-\sum_i \mathbf{r}_i \cdot \nabla_i V = V(\mathbf{r})$$

and

$$\ddot{I} = V + 2T \ .$$

(See Ostlie and Carroll pp. 53-56.)

If a gravitational system is in equilibrium, neither increasing or decreasing in size, it must have the long time average values $< V >$ and $< T >$ such that

$$< V + 2T > = 0 \quad , \quad < V > = -2 < T > \quad .$$

We can prove this by averaging over a long time Γ

$$\langle V + 2T \rangle = \frac{1}{\Gamma} \int_0^{\Gamma} (V + 2T) dt)$$

$$= \frac{1}{\Gamma} \left[\dot{I}(\Gamma) - \dot{I}(0) \right]$$

If Γ is the orbital period, $\dot{I}(T) = \dot{I}(0)$. More generally, if all particles remain bounded with bounded velocities for all time, *I(t)* remains bounded and the right-hand side tends to zero. (This relationship $<2T> = -< V>$ applies also to the kinetic and potential energies of many electron atomic systems bound by the Coulomb attraction between the nucleus and the electrons and can be established using quantum mechanics.)

(For a harmonic oscillator, $V \sim r^2$,

$$V(\lambda \mathbf{r}) \sim \lambda^2 V(\mathbf{r})$$

$$\frac{d}{d\lambda} V(\lambda \mathbf{r}) = 2\lambda V(\mathbf{r})$$

$$-\sum_i \mathbf{r}_i \nabla_i V = 2V(\mathbf{r})$$

$$\lambda = 1$$

$$\ddot{I} = -2V + 2T$$

$$<T> = <V> \quad)$$

For an alternative proof of the Virial Theorem see Ostlie and Carroll,pp. 53-56.

4.10 Gravitational Collapse

Imagine a cloud mass M, uniform density ρ, radius r_o

$$M = \frac{4}{3}\pi r_o^3 \rho$$

held at $r = r_o$ and released. In the absence of other forces, cloud will collapse. Conservation of energy

$$\frac{1}{2}\dot{\mathbf{r}}^2 = \frac{GM}{r} - \frac{GM}{r_o}$$

$$\therefore\ t_{ff} = \int_0^{t_{ff}} dt = -\int_0^{r_o} \left(\frac{dt}{dr}\right) dr$$

$$= \int_0^{r_o} \left[\frac{2GM}{r} - \frac{2GM}{r_o}\right]^{-1/2} dr \,.$$

Substitute $x = r/r_o$

$$t_{ff} = \left(\frac{r_o^3}{2GM}\right)^{1/2} \int_0^1 \left(\frac{x}{1-x}\right)^{1/2} dx \, .$$

Put $x = \sin^2\theta$; integral is $\pi/2$.

$$t_{ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}, \text{ depending only on } \rho \, .$$

Collapse time is independent of initial size.

For Sun, ρ=1.4 gm cm^{-3}

t_{ff} = 1.8 x 10^3 sec = 30 minutes.