4. Classical Dynamics

4.1 Newtonian Gravity

Two point masses M_1 and M_2 positioned at $\mathbf{r}_1$ and $\mathbf{r}_2$ attract one another. M_1 feels a force from $M_{2.}$

$$\mathbf{F}_{12} = \frac{-GM_1 M_2}{r_{12}^3}\mathbf{r}_{12}$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ is the vector from point 2 to point 1 and $r_{12} = |\mathbf{r}_{12}|$. The units of G are cm^3 s^{-2} g^{-1}. M_2 feels a force from M_1

$$\mathbf{F}_{21} = \frac{-GM_1 M_2}{r_{12}^3}\mathbf{r}_{21} = \frac{GM_1 M_2}{r_{12}^3}\mathbf{r}_{12} = -\mathbf{F}_{12}$$

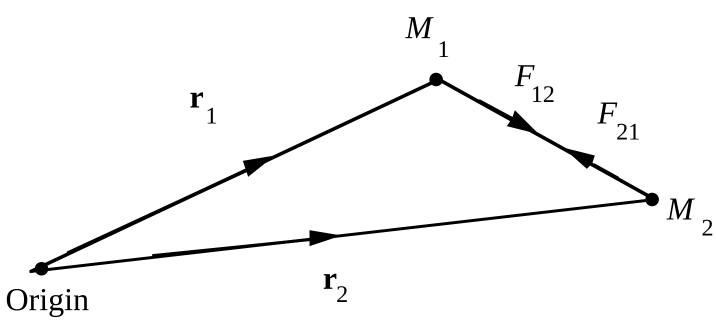


Fig. 4-1

Gravitational forces are equal and opposite in accord with Newton’s third law.

4.1.1 Newton's laws

1. A body remains at rest or in uniform motion unless acted on by a force
2. Force is equal to the time rate of change of momentum.
3. Action and reaction are equal in magnitudes and directly opposite in direction.

4.1.2 Gravitational potential

Gravitational forces can be represented by the gradient of a potential.

If $\mathbf{F}_{12}$ is the force on particle M_1 at $\mathbf{r}_1$ due to M_2 at $\mathbf{r}_2$

$$\mathbf{F}_{12} = -\nabla_1 V(\mathbf{r}_1)$$

where

$$V(\mathbf{r}_1) = \frac{-GM_1M_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

is the gravitational potential on a mass M_1 at $\mathbf{r}_1$ due to the presence of M_2 at $\mathbf{r}_2$ and

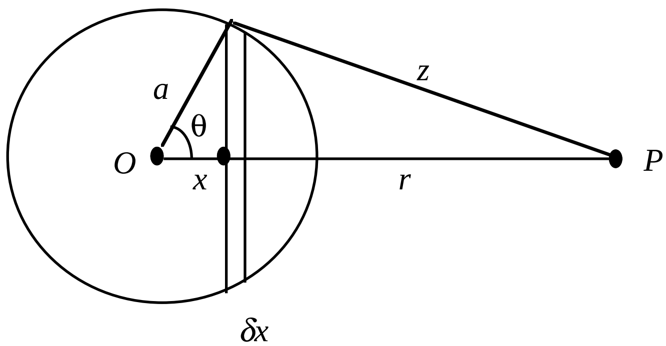
$$\nabla_1 = \left(\frac{\partial}{\partial x_1} , \frac{\partial}{\partial y_1} , \frac{\partial}{\partial z_1} \right) .$$

If there are N masses M_i at points $\mathbf{r}_i$, the potential for a test mass m at any point $\mathbf{r}$

$$V(\mathbf{r}) = -m \sum_{i=1}^{N} \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}$$

where the sum excludes any mass at **r**. The force is $-\nabla V(\mathbf{r})$.

4.1.3 Gravitational attraction of a spherical shell



Consider a thin spherical shell of mass *M* and radius a. *P* is a particle of mass *m* at a distance *r* from the center *O* of the shell and outside it. Divide the shell into rings x, $x + \delta x$ by planes perpendicular to *OP*.

Area of ring is $2\pi a \sin\theta \times a\delta\theta = 2\pi a\, \delta x$

where $x = a \cos\theta$, $\delta x = a \sin\theta\, \delta\theta$

Surface area of the shell is $4\pi a^2$.

So mass of ring is

$$2\pi a\delta x \ \frac{M}{4\pi a^2} \ = \ \frac{M\delta x}{2a}$$

Every point of the ring is equidistant by z from P so gravitational potential at P due to the ring is

$$d\text{V} = \frac{-Gm}{z} \quad \frac{M\delta x}{2a} \quad .$$

Total for the shell is

$$\text{V} = -\int_{-a}^{a} \frac{GmM}{2az}\,\delta x \,.$$

Now $z^2 = a^2 + r^2 \ - 2rx$

$2zdz = -\,2rdx$

$$\text{V} = \int_{r+a}^{r-a} \frac{GmM}{2ar}\,dz$$

$$= \frac{GmM}{2ar}\left[(r-a) - \ (r+a)\right] = -\frac{GmM}{r} \quad .$$

Force on the particle is the same as that exerted by a particle of mass equal to that of the spherical shell placed at the center of the shell.

If P is inside the shell,

$$V = \int_{a+r}^{a-r} \frac{GmM}{2ar}\, dz$$

$$= -\frac{GmM}{a}$$

which is a constant. So the force $\frac{-dV}{dr}$ vanishes—the shell exerts no force on particles inside it.

Ostlie and Carroll (pp. 36-38) give a similar discussion but use the force rather than the potential.

4.1.4 Solid sphere

Suppose the sphere is a solid with a mass distribution that is a function of r (or a constant). Add up the potentials of all the spherical shells—result is the same—gravitational potential on a particle outside a solid sphere is the same as that exerted by a particle having the mass of the sphere situated at its center.

Suppose particle is inside the solid sphere of mass M and radius R at a radius r. Shells with radii greater than r exert no force. Inside we have a solid sphere of mass $M(r)=Mr^3/R^3$ so gravitational force on the particle is

$$F(r) = \frac{GmM(r)}{r^2} = G\frac{m}{r^2}\frac{Mr^3}{R^3} = \frac{GmMr}{R^3} \quad .$$

4.1.5 Two solid spherical bodies

Force on each particle of sphere *A* is the same as produced by a particle of mass m_B at the center of sphere *B*. Add for all the particles. Gravitational force of *A* or *B* is the same as if the masses were particles at the centers of the two spheres.

4.2 The Two-body Problem

Equations of motion of two bodies at $\mathbf{r}_1'$ and $\mathbf{r}_2'$ with constant masses

$$M_1 \ddot{\mathbf{r}}_1' = \frac{-GM_1M_2}{|\mathbf{r}_1' - \mathbf{r}_2'|^3}(\mathbf{r}_1' - \mathbf{r}_2')$$

$$M_2 \ddot{\mathbf{r}}_2' = \frac{-GM_1M_2}{|\mathbf{r}_1' - \mathbf{r}_2'|^3}(\mathbf{r}_2' - \mathbf{r}_1') \ .$$

Add the two equations

$$M_1 \ddot{\mathbf{r}}_1' + M_2 \ddot{\mathbf{r}}_2' = 0$$

(linear momentum is conserved).

The center of mass is at the position $\mathbf{R}$ where

$$\mathbf{R} = \frac{M_1\mathbf{r}_1' + M_2\mathbf{r}_2'}{(M_1 + M_2)}$$

Then $\ddot{\mathbf{R}} = 0$, $\dot{\mathbf{R}}$ = constant = v_{cm} — in the absence of an external force, center of mass moves with uniform velocity.

Introduce coordinates $\mathbf{r}_1 = \mathbf{r}_1' - \mathbf{R}$

$\mathbf{r}_2 = \mathbf{r}_2' - \mathbf{R}$

$\mathbf{R} = \mathbf{R}_0 + v_{cm}t$ (t is the time)

relative to the center of mass

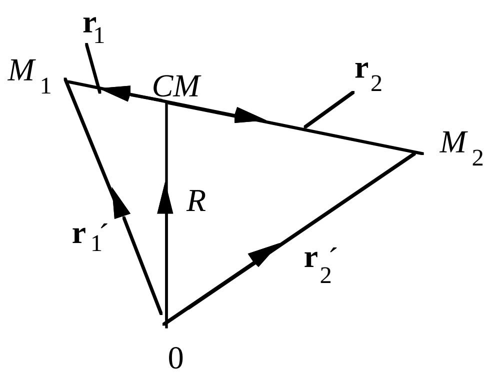


Fig. 4-2

Then

$$\mathbf{M}_1\ddot{\mathbf{r}}_1 = \frac{-GM_1M_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

$$\mathbf{M}_2\ddot{\mathbf{r}}_2 = \frac{-GM_1M_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_2 - \mathbf{r}_1) \ .$$

This independence of origin and velocity is called Galilean invariance. So choose origin as the position of the center of mass—i.e. take **R** = 0. (Then $\mathbf{r}_1 = \mathbf{r}_1'$ and $\mathbf{r}_2 = \mathbf{r}_2'$.) Now calculate the total angular momentum about origin.

$$\mathbf{L} = M_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + M_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 \quad .$$

For any central force

$$M_1 \ddot{\mathbf{r}}_1 = -\lambda \left(\mathbf{r}_1 - \mathbf{r}_2 \right)$$
$$M_2 \ddot{\mathbf{r}}_2 = -\lambda \left(\mathbf{r}_2 - \mathbf{r}_1 \right)$$

where for gravitation $\lambda = \dfrac{G M_1 M_2}{\left| \mathbf{r}_1 - \mathbf{r}_2 \right|^3}$.

Then

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= M_1 \left(\dot{\mathbf{r}}_1 \text{ x } \dot{\mathbf{r}}_1 + \mathbf{r}_1 \text{ x } \ddot{\mathbf{r}}_1 \right) + M_2 \left(\dot{\mathbf{r}}_2 \text{ x } \dot{\mathbf{r}}_2 + \mathbf{r}_2 \text{ x } \ddot{\mathbf{r}}_2 \right) \\ &= \left(\dot{\mathbf{r}}_1 \text{ x } M_1 \ddot{\mathbf{r}}_1 \right) + \left(\mathbf{r}_2 \text{ x } M_2 \ddot{\mathbf{r}}_2 \right) \\ &= -\lambda \left[\mathbf{r}_1 \text{ x } \left(\mathbf{r}_1 - \mathbf{r}_2 \right) + \mathbf{r}_2 \text{ x } \left(\mathbf{r}_2 - \mathbf{r}_1 \right) \right] \\ &= -\lambda \left[\mathbf{r}_1 \text{ x } \mathbf{r}_2 + \mathbf{r}_2 \text{ x } \mathbf{r}_1 \right] = 0 \ . \end{aligned}$$

L = constant vector, perpendicular to the plane of the motion and the motion is in a plane perpendicular to **L**.

We can also prove that the total energy $E = T + \mathrm{V}$ is constant where T is the kinetic energy and V the potential energy. Write the equations as

$$M_1 \ddot{\mathbf{r}}_1 = - \nabla_1 V, \quad M_2 \ddot{\mathbf{r}}_2 = - \nabla_2 V,$$

where V is the potential

$$V = \frac{-GM_1 M_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \quad .$$

Now using $\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} + \frac{df}{dz}\frac{dz}{dt}$ for any function f

we write

$$\frac{dV}{dt} = \nabla_1 V \cdot \dot{\mathbf{r}}_1 + \nabla_2 V \cdot \dot{\mathbf{r}}_2 \, .$$

Multiply equations of motion by $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ to give

$$M_1 \ddot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 = -\nabla_1 \mathbf{V} \cdot \dot{\mathbf{r}}_1$$

$$M_2 \ddot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 = -\nabla_2 \mathbf{V} \cdot \dot{\mathbf{r}}_2 \quad .$$

Thus adding the two equations,

$$\frac{d}{dt}\left[\frac{1}{2}M_1\dot{\mathbf{r}}_1^2+\frac{1}{2}M_2\dot{\mathbf{r}}_2^2+\mathrm{V}\right]=\frac{d}{dt}\left[T+\mathrm{V}\right]=0$$

i.e. $E_{tot} = T + \mathrm{V}$ = constant.

To describe a two-body system we need six functions of time $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ and twelve constants (say initial positions and momenta) must appear in the solutions.

We now have determined 3 for $\mathbf{R}$, 3 for $\mathbf{v}_{cm}$, 3 for $\mathbf{L}$ and 1 for E (these also apply to a many-body system). The two remaining are valid for two-body systems only.

4.2.1 Two-body orbits

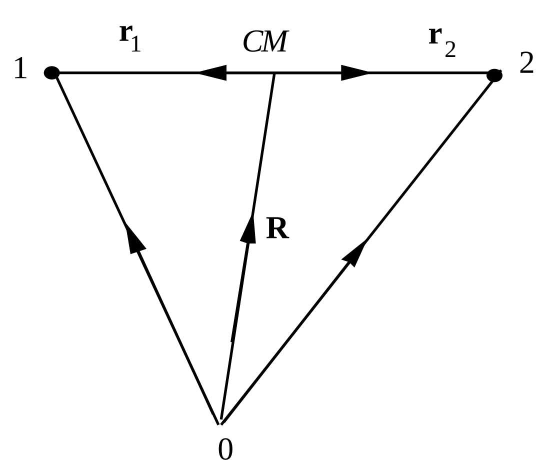


Fig. 4-3

$$\mathbf{R} = \frac{M_1\mathbf{r}_1 + M_2\mathbf{r}_2}{M} \ , \ M = M_1 + M_2$$

Introduce $\mathbf{r} = \left(\mathbf{r}_1 - \mathbf{r}_2\right)$

Then $\mathbf{r}_1 = \dfrac{M_2\mathbf{r}}{M}$, $\mathbf{r}_2 = \dfrac{-M_1}{M}\mathbf{r}$, $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$ **.**

Equations of motion of 1 and 2 are

$$M_1\ddot{\mathbf{r}}_1 = -\frac{GM_1M_2\ \mathbf{r}}{r^3}$$

$$M_2\ddot{\mathbf{r}}_2 = +\frac{GM_1M_2\ \mathbf{r}}{r^3} \quad .$$

Now consider the equation for the *relative* motion of the two particles

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2$$

$$= -\left(\frac{1}{M_1} + \frac{1}{M_2}\right)\frac{GM_1M_2\ \boldsymbol{r}}{r^3} = -\frac{1}{\mu}\frac{GM_1M_2\ \boldsymbol{r}}{r^3} \quad .$$

This can be written as mass × acceleration = gravitational force in the form

$$\mu\ddot{\mathbf{r}} = -\frac{G\mu M}{r^3}\mathbf{r}$$

where the *reduced mass* μ is given by

$$\frac{1}{\mu} = \frac{1}{M_1} + \frac{1}{M_2} \quad , \quad \mu = \frac{M_1 M_2}{M_1 + M_2} = \frac{M_1 M_2}{M} \; .$$

The equation describes the motion of a fictitious particle of mass μ under the gravitational attraction of a particle of mass $M = M_1 + M_2$, separated by **r**. The force can be written as the gradient with respect to **r** of a potential

$$V(r) = -\frac{GM_1 M_2}{r} = \frac{-G\mu M}{r} \; .$$

The total angular momentum of the system about the center of mass is

$$\mathbf{L} = \mathbf{r}_1 \times M_1\dot{\mathbf{r}}_1 + \mathbf{r}_2 \times M_2\dot{\mathbf{r}}_2 \; .$$

With

$$\mathbf{r}_1 = \frac{M_2\mathbf{r}}{M}\ ,\ \mathbf{r}_2 = -\frac{M_1\mathbf{r}}{M}$$

$$\mathbf{L} = \frac{M_1M_2}{M}\ \mathbf{r} \times \left(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2\right)$$

$$= \mu\mathbf{r} \times \dot{\mathbf{r}}\ .$$

Check:

$$\frac{d\mathbf{L}}{dt} = \mu\dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mu\mathbf{r}\times\ddot{\mathbf{r}}$$

$$= 0 + \mathbf{r} \times \left(\frac{-G\mu M}{r^3}\right)\mathbf{r} = 0\ .$$

The center of mass moves in a straight line and makes no contribution to the angular momentum ($\mathbf{R}\times\mathbf{v}_{cm} = 0$).

The angular momentum of the two-body system is identical to that of a single particle of mass μ moving with the relative velocities of the two bodies.

The kinetic energy of the two bodies is

$$T = \frac{1}{2} M_1 \dot{\mathbf{r}}_1{'}^2 + \frac{1}{2} M_2 \dot{\mathbf{r}}_2{'}^2$$

$$= \frac{1}{2} M_1 \left(\dot{\mathbf{r}}_1 + \dot{\mathbf{R}}\right)^2 + \frac{1}{2} M_2 \left(\dot{\mathbf{r}}_2 + \dot{\mathbf{R}}\right)^2$$

$$= \frac{1}{2} M_1 \left(\frac{M_2}{M}\right)^2 \dot{\mathbf{r}}^2 + \frac{1}{2} M_2 \left(\frac{M_1}{M}\right)^2 \dot{\mathbf{r}}^2$$

$$+ \dot{\mathbf{R}}\left(\frac{M_1 M_2}{M} \dot{\mathbf{r}} - \frac{M_2 M_1}{M} \dot{\mathbf{r}}\right)$$

$$+ \frac{1}{2} \left(M_1 + M_2\right) \dot{\mathbf{R}}^2$$

$$\therefore T = \frac{1}{2} \frac{M_1 M_2}{M} \dot{\mathbf{r}}^2 + \frac{1}{2} M \mathrm{v}_{cm}^2$$

$$= \frac{1}{2} \mu \dot{\mathbf{r}}^2 + \frac{1}{2} M \mathrm{v}_{\mathbf{cm}}^2 \,.$$

The kinetic energy of the two-body system is the kinetic energy of relative motion of a particle of mass μ plus the kinetic energy of the total mass M moving with the velocity of the center of mass. The potential energy is

$$\mathrm{V} = -\frac{G M_1 M_2}{r} = \frac{-G\mu M}{r} \,,$$

so the total energy of the system is the total energy of the particle of mass μ moving in the field of a particle of mass M plus the kinetic energy of mass M moving with the center of mass

$$E_{tot} = \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{1}{2}M\mathrm{v}_{cm}^{\;2} - \frac{G\mu M}{r}$$

$$= E + \frac{1}{2}M\mathrm{v}_{cm}^{\;2}.$$

E is the energy of relative motion.

In Cartesian coordinates, $\mathbf{L} = (0,\ 0, L)$, $\mathbf{r} = (\mathrm{x,y,0})$, $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, 0)$,

the angular momentum equation becomes

$$L = \mu(x\dot{y} - \mathrm{y}\dot{x})$$

and the energy equation is

$$E = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2\right) - \frac{G\mu M}{\left(x^2 + y^2\right)^{\frac{1}{2}}} \quad .$$

Better in polar coordinates

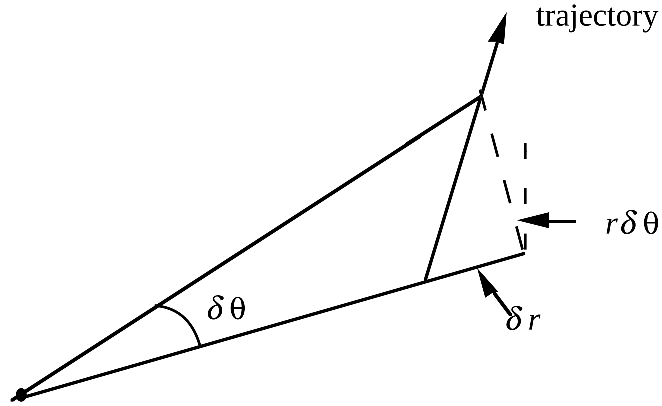


Fig. 4-4

or cylindrical coordinates (r, θ, z) with z perpendicular to the plane.

In time t,

$$\mathbf{r}(t) \to \mathbf{r}(t + \delta t)$$

$$\dot{\mathbf{r}} = \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t}$$

$$= \left(\frac{\delta r}{\delta t} \; , \; \frac{r\delta\theta}{\delta t} \; , \; 0 \right)$$

So components of the vector $\dot{\mathbf{r}} = \left(\dot{r} \; , \; r \; \dot{\theta} , \; 0 \right)$

Then

$$\mathbf{r} \times \dot{\mathbf{r}} = \left(0, \; 0, \; r^2 \dot{\theta} \right)$$

$$\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \dot{r}^2 + r^2 \dot{\theta}^2$$

So $L = \mu r^2 \dot{\theta}$,

where $\dot{\theta}$ = angular velocity

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{G\mu M}{r} \; .$$

Angular momentum equation can be interpreted geometrically.

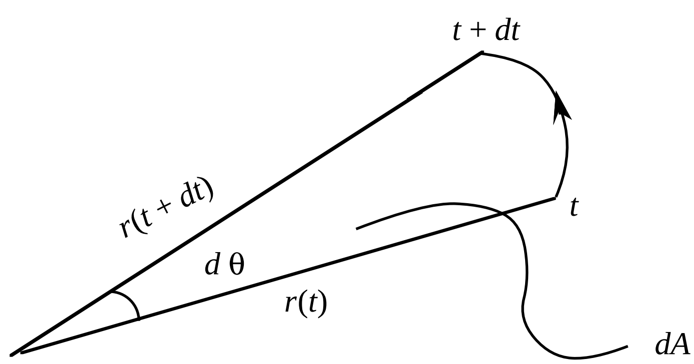


Fig. 4-5

In the interval *t*, *t* + *dt*, the orbit sweeps out an area *dA* and

$$dA = \frac{1}{2} r \text{ x } r d\theta$$

so

$$\frac{dA}{dt} = \frac{1}{2} r \text{ x } \frac{rd\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\therefore \quad \frac{dA}{dt} = \frac{L}{2\mu} = \text{constant} .$$

This is Kepler's second law. The radius vector to a planet sweeps out equal areas in equal intervals of time.

We can proceed further to obtain an equation in a single variable r. We have

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \qquad \text{(conservation of angular momentum)}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{1}{\mu}\right)^2 \frac{L^2}{r^2} \qquad \text{(conservation of energy).}$$

Write this as

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 - \frac{\mu GM}{r} + \frac{\mu}{2}\left(\frac{L}{\mu}\right)^2 \frac{1}{r^2} \quad .$$

The last term is the centrifugal repulsion. It is the opposite of the cenripetal force. It is an *inertial force* that arises to take account of the angular motion. We are in effect using a rotating frame with respect to which the particle has no angular motion and we need consider only radial motion. Consider a particle moving with constant angular velocity ω in a circle and ask what is the effective force. We showed in Chapter 1 that the force is $m\text{v}^2/r = \text{m}r\omega^2 = mr\,\dot{\theta}^2$. For a mass μ, the corresponding potential is $\frac{-1}{2}\ \mu\ r^2\ \dot{\theta}^2 = -\ \frac{\mu}{2}\ \left(\frac{L}{\mu}\right)^2 \frac{1}{r^2}$.

4.2.2 Runge-Lenz Vector (optional)

For a central force proportional to $1/r^2$, there is an additional conserved vector called the Runge-Lenz vector (though it was written down in 1799 by Laplace). The Runge-Lenz vector is

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu^2 GM\ \frac{\mathbf{r}}{r}$$

Then

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \dot{\mathbf{p}} \times \boldsymbol{L} = \mu\ddot{\mathbf{r}} \times \mathbf{L} = -\frac{\mu GM}{r^3}\ \mathbf{r} \times \mathbf{L}$$

$$= -\frac{\mu^2 GM}{r^3}\ \mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}).$$

Use vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - \dot{\mathbf{r}}\ r^2$$

$$= r^3 \left\{ \mathbf{r}\ \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3} - \frac{\dot{\mathbf{r}}}{r} \right\}.$$

$$\text{But } \mathbf{r}\cdot\dot{\mathbf{r}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{r}\cdot\mathbf{r}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(r^2) = r\dot{r}$$

and

$$\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{r}}) = r^3\left\{\frac{1}{r^2}\dot{r}\;\mathbf{r} - \frac{\dot{\mathbf{r}}}{r}\right\}.$$

Now $\frac{d}{dt}$ (unit vector) = $\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = -\frac{1}{r^2}\dot{r}\;\mathbf{r} + \dot{\mathbf{r}}/r$.

So

$$\mathbf{r}\times(\mathbf{r}\times\dot{\mathbf{r}}) = -r^3\frac{\mathrm{d}}{\mathrm{dt}}\hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \mathbf{r}/r.$$

$$\frac{d}{dt}(\mathbf{p}\times \mathrm{L}) = \frac{+\mu^2 GM}{r^3}r^3\frac{d\hat{\mathbf{r}}}{dt} = \mu^2 GM\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

$$\mathbf{p}\times\mathbf{L} - \frac{\mu^2 GM\mathbf{r}}{r} = \text{constant} = \mathbf{A}\text{, say}$$

Then $\mathbf{A}\cdot\mathbf{L} = 0$.

A is a fixed vector in the plane of the orbit.

4.2.3 Orbits

To find the trajectory as a function of time, we have to integrate the pair of conservation equations on page 4-18 with respect to time. But we can find the shape of the orbit (which is r as a function of θ)

$$\frac{dr}{d\theta} = \frac{dr}{dt} \Big/ \frac{d\theta}{dt}$$

$$= \frac{\mu}{L} r^2 \left[\frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{L}{\mu}\right)^2 \frac{1}{r^2} \right]^{1/2}$$

or

$$d\theta = \frac{(L/\mu)\,dr}{r^2 \left[\frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{L}{\mu}\right)^2 \frac{1}{r^2} \right]^{1/2}} \quad .$$

Integrate

$$\int_{\theta_0}^{\theta} d\theta = \theta - \theta_0 = \int^{r} \frac{(L/\mu)\,dr}{r^2 \left[\frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{L}{\mu}\right)^2 \frac{1}{r^2} \right]^{1/2}}$$

where θ_0 is a constant of integration.

Define a scale length by

$$r_o = \frac{(L/\mu)^2}{GM} = \frac{L^2}{GM\mu^2}$$

and a second constant ε by

$$\varepsilon^2 = 1 + \frac{2(E/\mu)(L/\mu)^2}{(GM)^2}$$

$$= 1 + 2EL^2/(GM)^2\mu^3 \ .$$

For a bound orbit, $\varepsilon < 1$ and E is negative. Then the integral can be written

$$\int^{r} \frac{r_o\,dr}{r^2\left\{\varepsilon^2 - \left(1 - r_o/r\right)^2\right\}^{1/2}} = \theta - \theta_0 \ .$$

Introduce a new variable u by

$$(1-r_o/r)^2 = \varepsilon^2 \cos^2 u$$

or

$$\varepsilon \cos u = \pm(1 - r_o/r) \ .$$

Then, adopting the + sign,

$$\varepsilon \sin du = -\frac{r_o\,dr}{r^2}$$

and

$$\frac{r_o\,dr}{r^2\left\{\varepsilon^2 - \left(1 - r_o/r\right)^2\right\}^{1/2}} = du.$$

We obtain

$$\int_o^u du = \theta - \theta_0$$

i.e. $u = \theta - \theta_0$, $\cos u = \cos(\theta - \theta_0)$

Hence

$$\varepsilon \cos(\theta - \theta_o) = 1 - r/r_o$$

or

$$\frac{1}{r} = \frac{1}{r_o}\left[1 \pm \varepsilon \cos\left(\theta - \theta_0\right)\right] .$$

We choose + and ε positive.

$$\frac{1}{r} = \frac{1}{r_o}\left\{1 + \varepsilon \cos\left(\theta - \theta_0\right)\right\}$$

where

$$r_o = L^3/GM\mu^2 \ , \ \varepsilon^2 = 1 + 2EL^2/(GM)^2\mu^3$$

is the orbit equation. It is the equation of a conic section. ε is called the *eccentricity*. There are three possibilities.

$\varepsilon = 0$ is a circle $r = r_o$

$\varepsilon < 1$ If $\varepsilon < 1$, r is always finite—the particles remain bound with

$$r_p = \frac{r_o}{1+\varepsilon} \le r \le \frac{r_o}{1-\varepsilon} = r_a$$

where r_p and r_a are the nearest and furthest parts of the trajectory from a focus, called respectively the perigee and the apogee. Then

$$\varepsilon = \frac{r_a - r_p}{r_a + r_p} \; .$$

$\varepsilon = 0$, the motion is a circle and the two foci coalesce at the center and $r_o = a$ is the radius. In describing the solar system, r_p and r_a are called, respectively, the perihelion and the aphelion.

Let us express the orbit in Cartesian coordinates.

$x = r \cos(\theta - \theta_0)$

$y = r \sin(\theta - \theta_0)$.

Then the orbit equation is

$$\frac{1}{\left(x^2 + y^2\right)^{1/2}} = \frac{1}{r_o}\left\{1 + \frac{\varepsilon x}{\left(x^2 + y^2\right)^{1/2}}\right\}$$

which is

$$\left(\frac{x+\varepsilon a}{a}\right)^2 + \frac{y^2}{b^2} = 1$$

where $a = \dfrac{r_o}{1-\varepsilon^2}$, $b = \dfrac{r_o}{\left(1-\varepsilon^2\right)^{1/2}}$.

Then

$r_p = a(1\text{-}\varepsilon)$, $r_a = a(1+\varepsilon)$.

a is the *semi-major* axis, b is the *semi-minor* axis, r_o is the *semi-latus* rectum.

The axial ratio is $\dfrac{b}{a} = (1\text{-}\varepsilon^2)^{1/2}$

a is the average Sun-planet distance which for Earth defines 1AU.

We choose θ_o =*O*. In the figure, *O* is origin of coordinates.

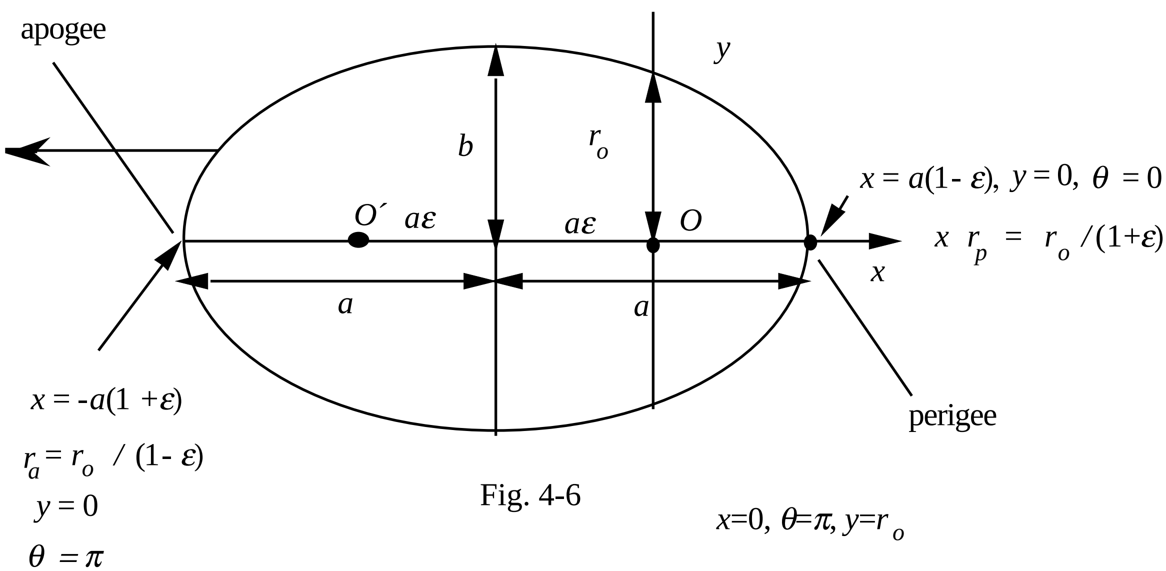


Fig. 4-6

b can be obtained from the coordinates of the point $x = O$, $y = r_0$

$$\frac{(a\varepsilon)^2}{a^2} + \frac{r_o^2}{b^2} = 1 \ , \quad \frac{r_o^2}{b^2} = 1 - \varepsilon^2$$

and

$$\frac{b^2}{a} = r_o \ .$$

The origin O of coordinates is the focus at the center of mass (close to the Sun). If the origin is taken at the center (the midpoint of the two foci) equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The standard definition (and how you can draw it) is the locus of the point P such that $r + r' =$ constant where r and r' are the distances from points O and O'.

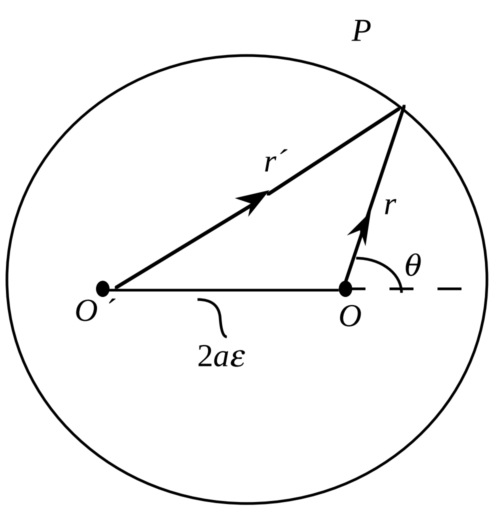


fig. 4-7

$$r'^2 = r^2 + (2a\varepsilon)^2 + 4a\varepsilon r \cos\theta$$

Orbit equation:

$$r = a(1-\varepsilon)^2/(1 + \varepsilon\cos\theta)$$

$$\therefore\ \varepsilon r\cos\theta = a(1-\varepsilon^2) - r$$

$$r'^2 = r^2 + 4a^2\varepsilon^2 + 4a^2(1-\varepsilon^2) - 4ar$$

$$= (r-2a)^2$$

$$\therefore\ r + r' = 2a.$$

$\underline{\varepsilon < 1}$ implies $E < 0$, the total energy of a bound system is negative.

The orbit is periodic and closed.

The period is obtained from $dt/d\theta = (d\theta/dt)^{-1}$,

$$\frac{d\theta}{dt} = \frac{(L/\mu)}{r^2}$$

using $\frac{1}{r} = \frac{1}{r_o}[1 + \varepsilon\cos(\theta)]$.

Hence $\int^t dt = \frac{r_o^2}{(L/\mu)} \int \frac{d\theta}{[1 + \varepsilon\cos(\theta)]^2}$.

In going around the orbit, $\theta \rightarrow \theta + 2\pi$

$$t \rightarrow t + \tau$$

where τ is the period so

$$\frac{r_o^2}{(L/\mu)} \int_0^{2\pi} \frac{d\theta}{[1 + \varepsilon \cos\theta]^2} = \tau.$$

Use the substitution $t = \tan(\theta/2)$

$$d\theta = \frac{2dt}{1 + t^2}, \; \cos\theta = \frac{1 - t^2}{1 + t^2} \quad .$$

Then

$$\tau = \frac{r_o^2}{(L/\mu)} \cdot \frac{2\pi}{(1 - \varepsilon^2)^{3/2}}$$

$$\text{or} \quad \tau = \frac{2\pi a^{3/2}}{(GM)^{1/2}} \quad .$$

The points at which the velocities are at right angles to the radius vector are called *apses*. The apse nearer to the Sun is the perihelion and the point further away is the aphelion (Fig. 4-6).

An alternative proof: integrate $\frac{dA}{dt}$ over the period P where A is the area of the orbit. Kepler's second law is

$$\frac{dA}{dt} = \frac{L}{2\mu} = \text{constant} . \quad (\text{cf. } 4-18)$$

So $A = \frac{LP}{2\mu}$ is the area of the ellipse and $P^2 = 4\mu^2 A^2/L^2$

But $A = \pi ab$

and $\dfrac{b^2}{a} = r_o = L^2 / GM\mu^2$.

$$\text{Then} \quad P^2 = \frac{4\mu^2 A^2}{L^2} = \frac{4\mu^2}{L^2}\pi^2 a^2 a \frac{L^2}{GM\mu^2}$$

$$P^2 = \frac{4\pi^2}{GM} a^3.$$

To obtain a simple formula for E, note that at perihelion r_p and aphelion r_a, $\dot{r} = 0$, so $\dot{\mathbf{r}}$ is perpendicular to $\mathbf{r}$ and

$$L = \mu v r = \mu v_p \, a\,(1-\varepsilon) = \mu v_a \, a\,(1+\varepsilon)$$

$$\frac{v_p}{v_a} = \frac{(1+\varepsilon)}{(1-\varepsilon)}$$

$$E = \frac{\mu v_p{}^2}{2} - \frac{GM\mu}{r_p} = \frac{\mu v_a{}^2}{2} - \frac{GM\mu}{r_a} .$$

Replace v_p by v_a $(1+\varepsilon)/(1-\varepsilon)$ and use $\dfrac{1}{r_a} - \dfrac{1}{r_p} = -\dfrac{2\varepsilon}{a(1-\varepsilon^2)}$

$$v_a = \sqrt{\frac{GM(1-\varepsilon)}{a(1+\varepsilon)}}$$

$$E = \frac{\mu GM(1-\varepsilon)}{2a(1+\varepsilon)} - \frac{\mu GM}{a(1+\varepsilon)}$$

$$= -\frac{GM\mu}{2a} \quad .$$

As $E \rightarrow 0$, $a \rightarrow \infty$.

From p. 4-22

$$L^2 = \frac{(1-\varepsilon^2)}{(-2E)}(GM)^2\mu^3$$

$$= (1-\varepsilon^2)GM\mu^2 a \quad .$$

$$L = \mu\left\{(1-\varepsilon^2)GMa\right\}^{1/2}$$

So $a = -\frac{GM_1M_2}{2E}$ depends only on energy, r_o depends only on angular momentum and ε depends on energy and angular momentum.

Kepler's Laws: for bound orbits,

1. the planets move in ellipses with the center of mass (the Sun) at one focus.
2. A line from the Sun sweeps out equal areas in equal times

$$\frac{dA}{dt} = \frac{1}{2}(L/\mu) \quad .$$

(*A* is the area here, not the magnitude of the Runge Lenz vector). $\frac{dA}{dt}$ does not depend on ε so the law applies also to unbound orbits with $\varepsilon \geq 1$.

3. The square of the period of revolution is proportional to the cube of the semi-major axis

$$\tau^2 = \frac{4\pi^2}{GM} a^3 \quad .$$

If we ignore the mass of the planet compared to the mass of the Sun, $M = M_\odot$, then, if τ is measured in years, call it the period *P*, and *a* is measured in AU,

$$P^2 = a^3 \ .$$

More generally, for a total mass M measured in $M_\odot$

$$P\ (\text{years})^2 = \frac{a\,(AU)^3}{M\,(M_\odot)} \quad .$$

Mean angular velocity $\omega = 2\pi/\tau$ so

$$\omega^2 = GM/a$$
.

To determine velocity at *r*, use conservation of energy

$$E = \frac{1}{2}\,\mu\,\dot{\mathbf{r}}^2 - \frac{G\mu M}{r} = \frac{-G\mu M}{2a} \quad .$$

Then

$$\dot{r}^2 = 2GM\left(\frac{1}{r} - \frac{1}{2a}\right) .$$

The closer to the focus, the faster the planet moves.

The relative motion is an ellipse. The actual bodies move in ellipses of the same shape but different sizes and all have the same angular velocity. Thus

$$\mathbf{r}_1 = \frac{M_2}{M}\mathbf{r}\ ,\ \dot{\mathbf{r}}_1 = \frac{M_2}{M}\dot{\mathbf{r}}$$

$$\mathbf{r}_2 = -\frac{M_1\mathbf{r}}{M}\ ,\ \dot{\mathbf{r}}_2 = -\frac{M_1\dot{\mathbf{r}}}{M}\ .$$

The Sun moves in a small orbit around the center of mass and the planet in a large orbit around the center of mass, always positioned so that they are on opposite sides of the center of mass. (It is this motion that is used to detect extrasolar planets.)

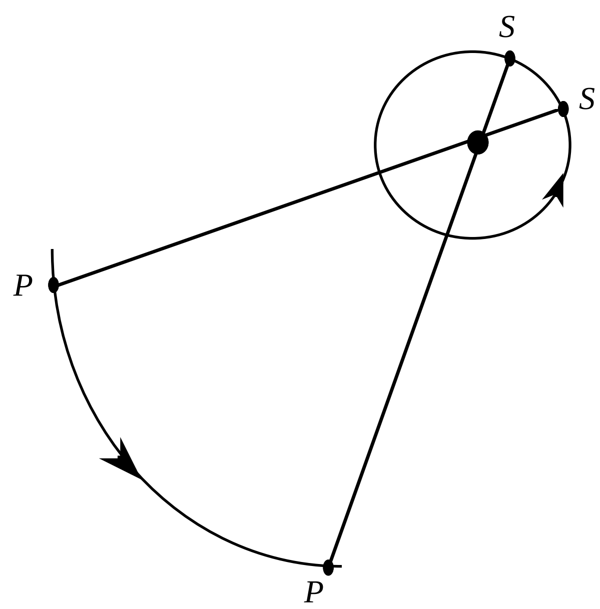


Fig. 4.8

Both orbits are ellipses.

Suppose we ignore the other planets and consider only Jupiter. The average Jupiter-Sun distance is 5.2 AU. The mass ratio of Jupiter to the Sun is 0.95×10^{-3}. The radius of the Sun's orbit is

$$\frac{M_J a}{M} = 0.95 \times 10^{-3} \times 5.2 \times 1.5 \times 10^8 \text{ km}$$

$$= 7.4 \times 10^5 \text{ km} .$$

The radius of the Sun is comparable at $R_\odot = 6.696 \times 105$ km.

<u>$\varepsilon > 1$</u> $\varepsilon = 1 + 2EL^2/(GM)^2\,\mu^3$ $E > 0$ and orbit is unbound - a hyperbola.

Write $\frac{1}{r} = \frac{1}{r_o}\left[1 + \varepsilon\cos(\theta - \theta_o)\right]$ into Cartesian coordinates

$$\frac{1}{\left(x^2 + y^2\right)^{1/2}} = \frac{1}{r_o}\left(1 + \frac{\varepsilon x}{\left(x^2 + y^2\right)^{1/2}}\right)$$

$$\therefore \quad \frac{\left(x - \varepsilon a\right)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\text{where } a = \frac{r_o}{\varepsilon^2 - 1}, \; b = \frac{r_o}{\left(\varepsilon^2 - 1\right)^{1/2}}$$

$$\frac{b}{a} = \left(\varepsilon^2 - 1\right)^{1/2} .$$

With origin at the midpoint, the equation for the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ .$$

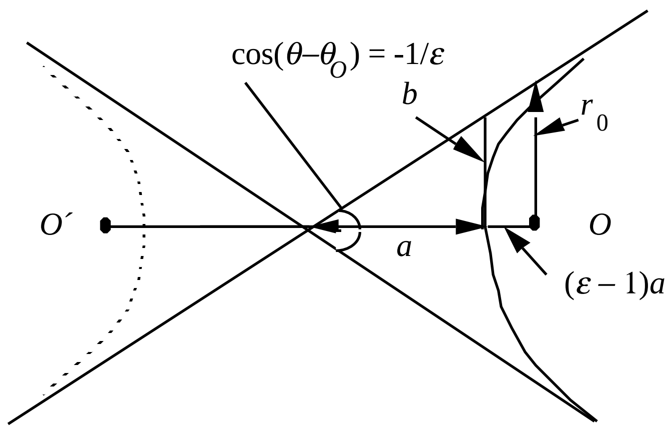


Fig. 4-9

A hyperbolic orbit

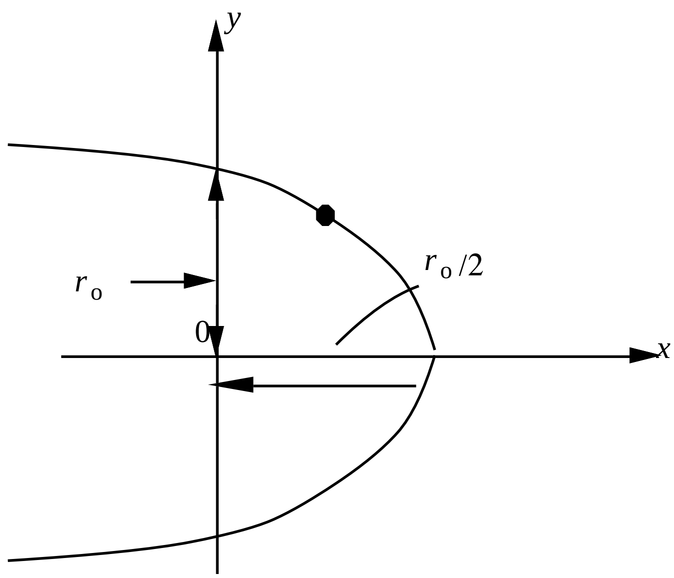
$\underline{\varepsilon = 1}$ $E = 0$, total energy is zero. Orbit is a parabola

$$y^2 = r_o^2 - 2r_o x$$

For a given distance r, $E = 0$ defines the escape velocity

$$\frac{1}{2}\mu v_{esc}^2 = \frac{G\mu M}{r} \quad , \quad v_{esc} = \sqrt{\frac{2GM}{r}} \ .$$

If $v \geq v_{esc}$, the particle escapes the gravitational field of *M*.



Parabola - single pass orbit

Fig. 4-10

If a particle of mass M_1 is moving in the gravitational field of a mass M_2, the particle escapes to infinity if at any r $v \geq v_{esc}$ where the escape velocity v_{esc}, corresponds to E=0. If $v \leq v_{esc}$, the particle is in a Keplerian orbit or is destined to crash into the origin.

4.2.4 Mass of Sun

The sidereal period of a planet, τ, (denoted earlier by P), is related to the semi-major axis

$$\tau^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

where

$$M = M_\odot + M_{Planet}$$

Measurements of τ

Planet	τ	a	GM (1026 cm3 s-1)	ε
Mercury	87.969	0.387099	1.32714	0.206
Venus	224.701	0.723332	1.32713	0.007
Earth	365.256	1.000000	1.32713	0.017
Mars	686.980	1.523691	1.32712	0.093
Jupiter	4332.589	5.202803	1.32839	0.048
Saturn	10759.22	9.53884	1.32750	0.056
Uranus	30685.4	19.1819	1.32715	0.047
Neptune	60189	30.0578	1.32723	0.009
Pluto	90465	39.44	1.32727	0.249

Given τ and a, we can obtain $M_p + M_\odot$.

From low mass planets

$$G\, M_\odot \;= 1.32713 \text{ x } 10^{26} \text{ cm}^3 \text{ s}^{-2}.$$

Now

$$G = 6.674215 \times 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1} \;.$$

Then $M_\odot = 1.988435 \times 10^{33}$g.

We can also derive mass of Jupiter.

$$G(M_\odot + M_J) = 1.32839\ 10^{26} \text{ cm}^3 \text{ s}^{-1}$$

$$G(M_\odot + M_E) = 1.32713\ 10^{26} \text{ cm}^3 \text{ s}^{-1}$$

$$\therefore\ G(M_J - M_E) = 1.26\ 10^{23} \text{ cm}^3 \text{ s}^{-1}$$

$$\frac{M_J - M_E}{M_\odot} = 0.000949 \;.$$

$$M_J \sim 0.000949\, M_\odot$$

$$= 1.89 \text{ x } 10^{30} \text{ g}$$

Better estimates can be made from the orbits of planetary satellites and spacecraft.

4.2.5 Interplanetary travel

Spacecrafts travel in orbits around the Sun. Suppose a spacecraft is directed to Mercury. We wish to place it in an orbit around the Sun that is tangent to the Earth at aphelion and tangent to Mercury at perihelion.

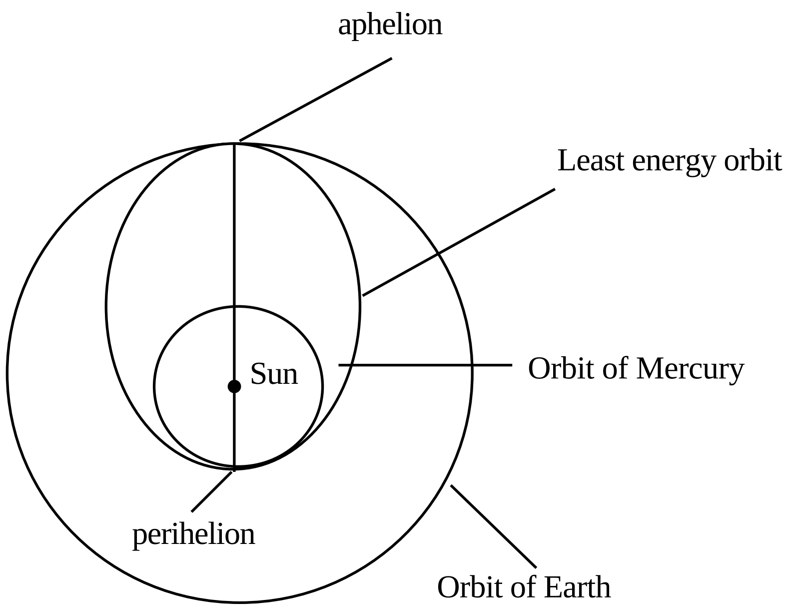


Fig. 4-11

That orbit has the smallest a and therefore takes the least energy. Assume orbits of Earth and Mercury are circular. Major axis is the sum of the aphelion and perihelion distances

$$2a = 0.387 + 1.000 = 1.387 \text{ AU}$$

$$a = 0.694 \text{ AU} = 1.04 \times 10^{11} \text{ m.}$$

The initial orbital speed at aphelion comes from the conservation of energy

$$v_a^2 = GM_\odot \left(\frac{2}{r_a} - \frac{1}{a} \right)$$

(We are ignoring the gravitational fields of the Earth and Mercury) Then with r = 1.496×10^{11} m,

$$M_\odot = 1.988 \times 10^{30} \text{ kg}$$

$$v_a = 22 \text{ km s}^{-1} .$$

Each is orbiting the Sun at 30 km s^{-1} , so we launch at 8 km s^{-1} in a direction *opposite* to the direction of the Earth's motion.

4.2.6 Moment of inertia of a spinning sphere

The angular momentum of a particle of mass m orbiting about a center with angular velocity ω is

$$L = mr^2\dot{\theta} = mr^2\omega = I\omega$$

and its rotational kinetic energy is

$$T = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2} I\omega^2 .$$

The angular momentum of a spherical body rotating about an axis with angular velocity ω is similarly $I\omega$ and the kinetic energy is $\frac{1}{2} I\omega^2$ where I is called the moment of inertia. For a uniform sphere of mass M and radius R,