AY208: Meeting #7

Last Time Radiative Transfer "Review" & Einstein Coefficients

Today:
1. Absorption & Emission in Terms of Einstein Coeff.
2. Radiative Transfer in Terms of Einstein Coeff.

End of Last Time.

$$\boxed{\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu} \quad (12)$$

where $\boxed{S_\nu \equiv \frac{j_\nu}{\alpha_\nu}}$ (13) = source funxn

emission coeff (j_ν); absorption coeff (α_ν)

$(\alpha_\nu = \rho \kappa_\nu)$, κ_ν = opacity

$$\boxed{\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')\,ds' = \underbrace{\alpha_\nu s}_{\text{for } T \text{ indep of } s}} \quad (11b)$$

$\int (12) \Rightarrow$

$$\boxed{I_\nu(\tau_\nu) = \underbrace{I_\nu(0)}_{\text{orig. int.}}\, \underbrace{e^{-\tau_\nu}}_{\text{abs}} + \int_0^{\tau_\nu} \underbrace{e^{-(\tau_\nu-\tau_\nu')}}_{\text{abs term}}\, \underbrace{S_\nu(\tau_\nu')}_{\text{source}}\, d\tau_\nu'} \quad (14)$$

NEW

If S_ν is independent of τ_ν' (e.g. a blob of uniform comps'n, T, n) **then**

$$\boxed{\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu\left(1-e^{-\tau_\nu}\right) \\ &= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \end{aligned}} \quad (15)$$

called "constant source funxn" approx'n

Interesting Limits
- $\tau > 1$ optically thick, opaque
- $\tau < 1$ " thin, transparent

and very thick $\tau_\nu \to \infty$ $\quad I_\nu \to S_\nu$ (e.g. a rock)

very thin $\tau_\nu \to 0$ $\quad I_\nu(\tau_\nu) \to I_\nu(0) + S_\nu \tau_\nu$ (15a)

since $e^{-\tau_\nu} \to 1 - \tau_\nu$ for small τ

Think for a minute.
What does all this mean?

(a) 5 ⇒ For spontaneous emission alone intensity increases ($I_\nu \propto \int j_\nu ds$) ~ linearly

(a) 6 ⇒ For absorption alone intensity decreases exponentially
$I_\nu \propto I_\nu(0) e^{-\tau}$.

If τ is large then specific value doesn't matter $I_\nu \rightarrow S_\nu$

If τ is small then $e^{-\tau} \rightarrow 1-\tau$ and exponential behavior → linear behavior

---

Now let's connect ~~[illegible]~~ formalism to previous discussions

Thermo. Eq

Einstein A, B coeff

masers — more from LJG

then we'll add scattering...

THERMAL RADIATION
(Emitting & absorbing material in T.E.)

$\boxed{S_\nu = B_\nu(T)}$ or, using 13, $\boxed{j_\nu = \alpha_\nu B_\nu(T)}$ (16)

emission, absorption

"Kirchoff's Law"

note: this is not the same as saying
$I_\nu = B_\nu(T)$, which is only true (see 15a) when $\tau \rightarrow \infty$

"Thermal radiation becomes blackbody radiation only for optically thick media."

In general (re-writing 15) for thermal radiation @ one T

$$\boxed{I_\nu = I_\nu(0)e^{-\tau_\nu} + B_\nu(T)\left[1-e^{-\tau_\nu}\right]} \quad (17)$$

$I_\nu(0) \rightsquigarrow \,|\; \tau \;|\, \rightsquigarrow I_\nu$

isothermal stuff

Actually, we assumed one T for whole emitting region to write (17), more generally T can be, in essence, $T(\tau)$ which makes integ more complicated

---

Recall from meeting #4: Brightness Temperature, T_B

$$\boxed{I_\nu = B_\nu(T_B) = \frac{2h\nu^3}{c^2}\,\frac{1}{\exp\left(\frac{h\nu}{kT_B}\right)-1}} \quad (18)$$

by definition of T_B

radiate xfer can be expressed as propagation of T_B in R-J regime; because $I_\nu^{RJ} \propto T_B$, as follows:

Recall

for $h\nu \ll kT$ (18) reduces to $\boxed{I_\nu^{RJ}(T) = \frac{2\nu^2}{c^2}\,kT}$ (19)

$I_\nu \propto \nu^2 T$ ← absurd!! eg opt thick HII reg

"Rayleigh-Jeans limit"

for $h\nu \gg kT$ $\boxed{I_\nu^{W} = \frac{2h\nu^3}{c^2}\exp\left(\frac{-h\nu}{kT}\right)}$ (20)

"Wien limit"

So, using (19) eq. (17) becomes:

R-J only

$$\boxed{T_B = T_B(0)\,e^{-\tau_\nu} + T\left(1-e^{-\tau_\nu}\right)} \quad (21)$$

useful in radio astro

note: T_B does not depend on ν for a true black body — by definition

as $\tau \to \infty$ $T_B \to$ true temperature of the material

Absorption & Emission Coefficients in Terms of Einstein Coeff

Spontaneous Emission: j_ν

→ each transition produces $h\nu$ distributed 4π steradians

(Recall: A_{ul} = trans prob per unit time)

$$\therefore \quad \boxed{j_\nu = \frac{h\nu}{4\pi} n_u A_{ul} \phi(\nu)} \quad (22)$$

Absorption: α_ν by similar reasoning

$$\boxed{\alpha_\nu = \frac{h\nu}{4\pi} n_l B_{lu} \phi(\nu)} \quad (23)$$

(uncorrected for stimulated emission)

Stimulated Emission: ~ "Negative Absorption" (depends on $\bar{J}$ as does Abs)

So this adds a " $-\frac{h\nu}{4\pi} n_u B_{ul} \phi(\nu)$ " term to 23, giving

Total absorption coefficient:

$$\boxed{\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) \left(n_l B_{lu} - n_u B_{ul} \right)} \quad (24)$$

Abs   stim Em

Radiative Transfer in Terms of Einstein Coefficients

Eq. $\boxed{10}$ (22), (24) give

becomes

transfer eqn

$$\boxed{\frac{dI_\nu}{ds} = \underbrace{\frac{-h\nu}{4\pi}\left(n_\ell B_{\ell u} - n_u B_{u\ell}\right)\phi(\nu)}_{\alpha_\nu} I_\nu + \underbrace{\frac{h\nu}{4\pi} n_u A_{u\ell}\,\phi(\nu)}_{j_\nu}} \quad (25)$$

Recall that $S_\nu = \frac{j_\nu}{\alpha_\nu}$ (13), so, using (22) & (24) gives

$$\boxed{S_\nu = \frac{n_u A_{u\ell}}{n_\ell B_{\ell u} - n_u B_{u\ell}}} \quad (26)$$

Remember Einstein relations

$$\boxed{\begin{aligned} g_\ell B_{\ell u} &= g_u B_{u\ell} \\ A_{u\ell} &= \frac{2h\nu^3}{c^2} B_{u\ell} \end{aligned}} \quad (27)$$

only need one of $A_{u\ell}$, $B_{u\ell}$, $B_{\ell u}$ to get other two

So we can write α_ν & S_ν in terms of just $B_{\ell u}$ & stat wts:

(24) $\Rightarrow$
$$\boxed{\alpha_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u}\left(1 - \frac{g_\ell n_u}{g_u n_\ell}\right)\phi(\nu)} \quad (28)$$

(26) $\Rightarrow$
$$\boxed{S_\nu = \frac{2h\nu^3}{c^2}\left(\frac{g_u n_\ell}{g_\ell n_u} - 1\right)^{-1}} \quad (29)$$

& we get a VERY GENERAL Q.M. description of absorption & emiss proc that can be propagated using $\int$ (10) to give "macroscopic" results!!

"Examples" (Interesting cases of eqs 28, 29)

Ⓐ "LTE"

$$\boxed{\frac{n_\ell}{n_u} = \frac{g_\ell}{g_u}\exp\left(\frac{h\nu}{kT}\right)} \quad (30)$$

locally holds

28 ⟹ $$\boxed{\alpha_\nu = \frac{h\nu}{4\pi} n_\ell B_{\ell u}\left[1-\exp\left(\frac{-h\nu}{kT}\right)\right]\phi(\nu)} \quad (31)$$

29 ⟹ $$\boxed{S_\nu = B_\nu(T)} \quad \text{(Kirchoff's Law)} \quad (32)$$

(gives back eq 16)

$= \frac{j_\nu}{\alpha_\nu}$ (Definition of)

abs & em are simply related since Th. Eq.

just need one Einstein coeff, line prof, & T

Ⓑ "Non-Thermal Emission" $\frac{n_\ell}{n_u} \neq \frac{g_\ell}{g_u}\exp\left(\frac{h\nu}{kT}\right)$

eg Synchrotron, scattering present, etc

More later in course!

Ⓒ "Inverted Populations & Masers"

Inverted Masing

causes: $\frac{n_\ell}{n_u} < \frac{g_\ell}{g_u}$ (33)

usually $g_\ell < g_u$ so ⟹ $n_\ell << n_u$

⟹ much emission

⟹

$\alpha_\nu < \emptyset$ intensity increases along a ray!

$\tau < 0$

e.g. $\tau = -100$

amplification 10^{43}

(Normally:) $\frac{n_u g_\ell}{n_\ell g_u} = \exp\left(\frac{-h\nu}{kT}\right) < 1$ $\quad e^{-x} \to 1-x$

which means $\frac{n_\ell}{n_u} > \frac{g_\ell}{g_u}$

& $\frac{g_\ell}{g_u}$ usually <1 so $\frac{n_\ell}{n_u} > 1$

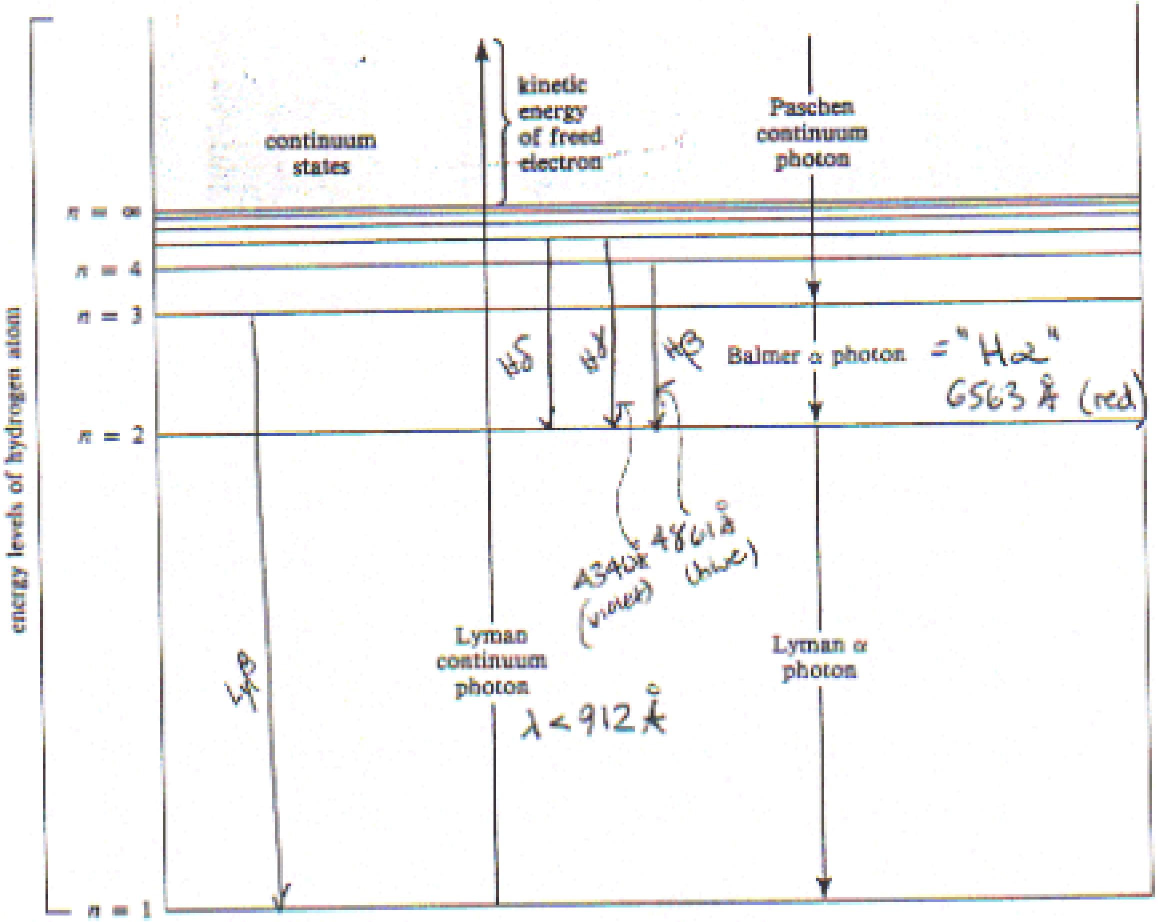

Figure 3.20. Radiative transitions to and from the continuum. Bound states of hydrogen are represented by the horizontal lines labeled $n = 1, n = 2, \ldots, n = \infty$, with the height from the ground state ($n = 1$) proportional to the energy difference of the states. Above the last bound state, $n = \infty$, exists a continuum of free states (ionized hydrogen). An electron can be freed into the continuum from the ground state by the absorption of a Lyman continuum photon. Conversely, a free electron may be captured, say, into the level $n = 3$ with the emission of a Paschen continuum photon. The electron in $n = 3$ may then cascade to the ground level by the subsequent emission, say, of a Balmer α photon and a Lyman α photon.

Note:
Recombination lines, e.g. H109α @ 6cm occur btwn very high bound states (e.g. n= 109).

emission or in absorption, would constitute spectroscopic evidence for the presence of hydrogen.

Problem 3.7. For atomic hydrogen show that equation (3.8) results in the expression

$$\lambda = 4\pi\left(\frac{\hbar c}{e^2}\right)\left(\frac{n^2\hbar^2}{m_e e^2}\right)\left(\frac{n'^2}{n'^2 - n^2}\right),$$

where n' and n are, respectively, the principal quantum numbers associated with the upper and lower levels. The combination $\hbar c/e^2$ is known as the inverse of the "fine-structure constant" and has an approximate value of 137. The combination $n^2\hbar^2/m_e e^2$ is the size of the hydrogen atom in the lower state. Thus, if the last term is of order unity, the hydrogen atom typically emits and absorbs radiation with wavelength roughly 10^3 times its own size. (This fact leads to an important approximate method for calculating radiation probabilities, but it lies outside

Shu, The Physical Universe, 1982