Astronomy 208 Meeting #6 (+)

(2.3a)ₐ

① – The Very Basics of Radiative Transfer (Part I: Definitions)

② – Einstein Coefficients (2.3b)

③ – Introductory Radiative Transfer (Part II)

④ – Absorption & Emission in Terms of Einstein Coefficients

⑤ Radiative Transfer in Terms of Einstein Coefficients

"1" Radiative Transfer Definitions

(enough to use J_ν, the mean intensity, in discussing
Einstein coefficients)
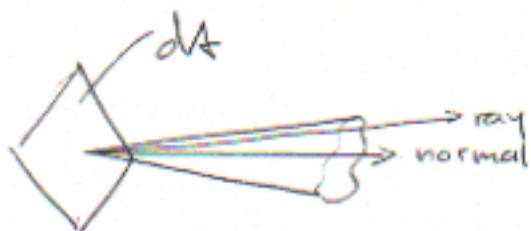
For this example:
differentially small $d\Omega$
"normally incident rays"
w/in $d\Omega$ of normal
(more general expressions later)
(Projected area ≡ real area)

$$\boxed{\begin{array}{l} dE = \text{energy crossing } dA \text{ in a time } dt, \text{ w/in freq range } d\nu \\ \quad = I_\nu \, dA \, dt \, d\Omega \, d\nu \end{array}} \quad (R1)$$

(R2) I_ν = specific intensity (distance independent) erg s⁻¹ ster⁻¹ cm⁻² Hz⁻¹
along a ray $dI_\nu/ds = 0$ in free space
note – not per $d\Omega$

(R3) $F_\nu = \int_{4\pi} I_\nu \, d\Omega$ = Flux   erg s⁻¹ cm⁻² Hz⁻¹

(R4) $u_\nu = \frac{1}{c} \int_{4\pi} I_\nu \, d\Omega$ = Energy Density   erg cm⁻³ Hz⁻¹ $(= \frac{4\pi}{c} J_\nu)$

(R5) $J_\nu = \frac{1}{4\pi} \int_{4\pi} I_\nu \, d\Omega$ = Mean Intensity   erg s⁻¹ cm⁻² Hz⁻¹
Same units as F_ν

Re-write Xfer Equ (10) as:

$$\boxed{\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu} \quad \text{(12)}$$

where $\boxed{S_\nu \equiv \frac{j_\nu}{\alpha_\nu}}$ = "source function" (13)

(emission) / (absorption (incl. stim. em.))

note $d\tau$ is a more natural
"line-of-sight" unit than ds

So then, integrating ⑫ gives soln to Xfer equ:

$$\boxed{I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu')d\tau_\nu'} \quad \text{④}$$

initial intensity diminished by absorption (see picture next page)

Integrated source function along depth τ_ν from obsv'r, diminished by absp'n

τ_ν' is dummy variable so that we can allow S_ν to depend on $\tau_\nu^{(1)}$

Note | $\tau_\nu > 1$ | optically thick (opaque) |
| $\tau_\nu < 1$ | optically thin (transparent) |

If S_ν is independent of τ_ν' (e.g. a blob of uniform comp, T, n) then

"constant source function"

$$④ \Rightarrow \boxed{\begin{aligned} I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \\ &= S_\nu + e^{-\tau_\nu}(I_\nu(0) - S_\nu) \end{aligned}} \quad \text{⑤}$$

Interesting limits | $\tau_\nu \to \infty \quad I_\nu \to S_\nu$ (e.g. a rock)
$\tau_\nu \to 0 \quad I_\nu(\tau_\nu) \to I_\nu(0) + S_\nu \tau_\nu$ ⑤a

recall $e^{-x} \to 1 - x$ for small x & $\tau_\nu I_\nu(0) \ll I_\nu(0)$

Some Important Definitions/Equations to Remember from Rad Xfer

j_ν = emission coeff = $\frac{\epsilon_\nu \rho}{4\pi}$ ϵ_ν = emissivity

α_ν = absorption coeff = $\rho \kappa_\nu$ κ_ν = opacity

$$\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')ds' = \alpha_\nu s = \text{optical depth}$$
$$\text{(for } \alpha_\nu \text{ indep of } s\text{)}$$

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu')d\tau_\nu'$$

$$S_\nu \equiv j_\nu/\alpha_\nu$$

If S_ν indep of τ_ν'
$$\left[\begin{array}{l} \tau_\nu \Rightarrow \infty \quad I_\nu \to S_\nu \\ \tau_\nu \to 0 \quad I_\nu(\tau_\nu) \to I_\nu(0) + S_\nu \tau_\nu \end{array} \right]$$

For Thermal Radiation:

Rayleigh-Jeans Limit: $h\nu \ll kT$ $I_\nu^{RJ} = \frac{2\nu^2}{c^2}kT$ $(I_\nu \propto \nu^2 T)$

Wien Limit $h\nu \gg kT$ $I_\nu^{W} = \frac{2h\nu^3}{c^2}\exp\left(\frac{-h\nu}{kT}\right)$

$$\left[\begin{array}{l} \text{for R-J case} \quad T_B = T_B(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu}) \\ \quad \text{as } \tau \to \infty \quad T_B \to \text{true temp of material} \\ \quad \text{as } \tau \to 0 \quad T_B \to T_B(0) + T\tau_\nu \end{array} \right]$$

A note on statistical weights

Full specification of an atomic energy level involves four quantum #'s : n, ℓ, s, m

- electronic or "principal"
- orbital quantum #
- spin ang mom
- nuclear of $j = \ell + s$
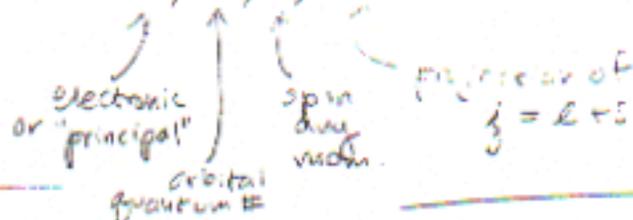
$\Big\{$ Note: mg mom of $\vec{e}$ — orbital mg mom of atom   spin-orbit   fine struct
mg mom of e^- - spin of proton   spin-spin   hfs $\Big\}$

e.g. For H with $\downarrow$

$n=2$ —
  $\ell=1$ <
    $j = 3/2$ $m = -3/2, -1/2, 1/2, 3/2$ $\Big\}$
    $j = 1/2$ $m = -1/2, 1/2$
  $\ell=0$ $j = 1/2$ $m = -1/2, 1/2$

$\Big\}$ 8-fold degeneracy
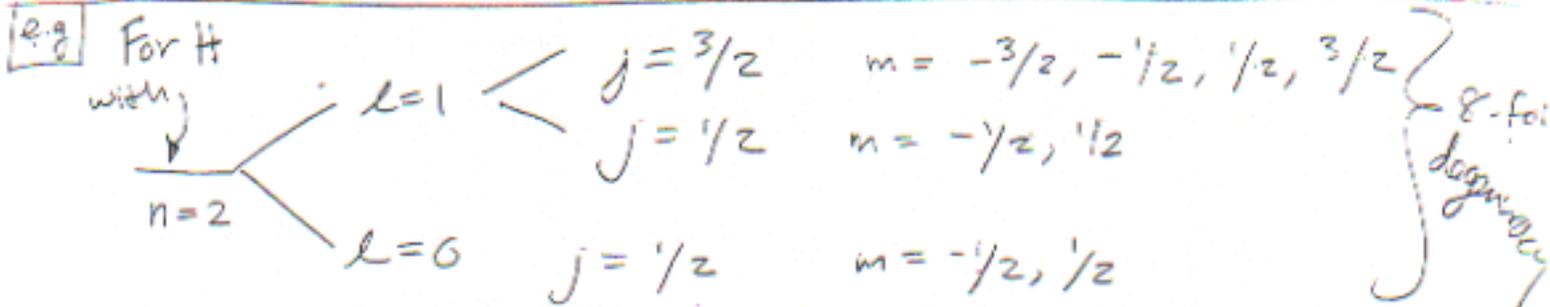
Number of sublevels of principal quantum number n
usually $= \boxed{2n^2}$ = statistical weight $= \boxed{g_n}$
= "number of distinct quantum states contained in level n"

What about molecules, ions?   "g" is still ~ # of available sub-levels (ie degeneracy) but a bit more complex to calculate

— Einstein Coefficients —

"1" Einstein's Motivation in 1917

In 1901 Planck found expression for BB

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \left[\exp\left(\frac{h\nu}{kT}\right) - 1 \right]^{-1}$$

recall this is for photons in thermodynamic equilibrium

Previously (1894) there was just Wien's Law

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \exp\left(\frac{-h\nu}{kT}\right)$$

which applies when $h\nu \gg kT$ and is derived semi-classically $\left(\frac{h\nu}{kT} = \frac{E}{kT}\right.$, a Boltzmann factor$\left.\right)$

GOAL: Quantum processes, at least in equilibrium, must give rise to the <u>same</u> distribution of 'atoms' over their quantum states as predicted by general laws of thermodynamics.

<u>Einstein's approach</u>: match <u>Planck</u> law by assigning transition <u>probabilities</u> to absorptive & emissive processes.

<u>Note</u>: If only "ordinary" (abs) & (spontaneous emission) are included one can only get <u>Wien Law</u>.

Einstein needed to include <u>stimulated emission</u>.
(Prob Set)

Note:

$$\Delta E_{ul} = h\nu_{ul} \neq h\nu_0$$

Absorption
B_{lu}
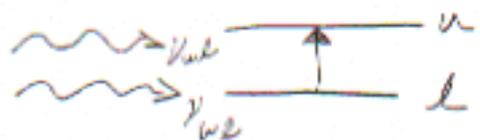
$\rightsquigarrow \nu_{ul}$ ___ u $\rightsquigarrow \nu_{ul}$ (one photon removed)
$\rightsquigarrow \nu_{ul}$ ___ l

Spontaneous Emission
A_{ul}

___ u $\rightsquigarrow \nu_{ul}$ (one photon produced)
___ l

Stimulated Emission
B_{ul}
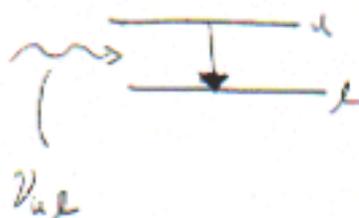
$\rightsquigarrow$ ___ u $\rightsquigarrow \nu_{ul}$ (one photon produced)
$\rightsquigarrow \nu_{ul}$ ___ l $\rightsquigarrow \nu_{ul}$

ν_{ul}

Note: Equ #'s in what follows continue from Meeting 5, as they form a "unit."

Einstein Coefficients $A_{u\ell}$, $B_{\ell u}$
(see Figure on prev. page)

$\Delta \mathcal{E} = h\nu_0$

(i)
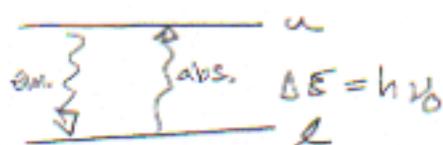
$A_{u\ell} \equiv$ transition probability per unit time for spontaneous emission (sec^{-1})

(ii)

$\boxed{B_{\ell u} \bar{J} =}$ transition prob. per unit time for absorption (sec^{-1})

where

⑩ $\boxed{\bar{J} = \int_0^\infty \phi_\nu J_\nu \, d\nu = \int_0^\infty \phi_\nu \frac{1}{4\pi} \oint I_\nu \, d\Omega \, d\nu}$ (from e5)

= line-profile-integrated mean intensity

and

$\phi_\nu =$ line-profile shape, normalized so that $\boxed{\int_0^\infty \phi_\nu \, d\nu = 1}$ ⑪
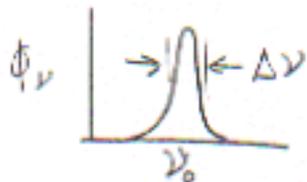
ϕ_ν $\rightarrow\!|\!\leftarrow \Delta\nu$ (We'll return to line-profile shapes later...)

ν_0

(iii) $B_{u\ell} \bar{J} =$ transition prob per unit time for stimulated emission
Einstein found necessary to get Planck funxn back from all raxus (see below)

note: If J_ν changes slowly over $\Delta\nu$, then one can
approximate $\phi(\nu)$ as a δ funxn. so that

(ii) $\rightarrow B_{\ell u} J_{\nu_0}$ and (iii) $\rightarrow B_{u\ell} J_{\nu_0}$

(Einstein did this implicitly)

Relations between Einstein Coefficients (in Thermodynamic Eq)

"detailed balance"   # trans into state i = # trans out of state i

$n_u =$ } number density in u, ℓ state
$n_\ell =$ }

$$\boxed{n_\ell B_{\ell u} \bar{J} = n_u A_{u\ell} + n_u B_{u\ell} \bar{J}} \quad ⑫$$

$\underline{\text{(absorption)}} = \overline{\text{(spont em} + \text{stim em)}}$

Solving for J from ⑫

$$\boxed{\bar{J} = \left(\frac{n_u A_{u\ell}}{n_\ell B_{\ell u} - n_u B_{u\ell}} \right)^{\text{skip}} = \frac{A_{u\ell}/B_{u\ell}}{(n_\ell/n_u)\left(\frac{B_{\ell u}}{B_{u\ell}}\right) - 1}} \quad ⑬$$

Recall from Meeting #4 eq ④ ; in Th.Eq. $\frac{n_\ell}{n_u}$ given by

$$\boxed{\frac{n_\ell}{n_u} = \frac{g_\ell}{g_u} \exp\left(h\nu_0/kT \right)} \quad ⑭$$

(Boltzmann eqn.)

↙ see handout on "g"s

so then ⑬ becomes

$$\boxed{\bar{J} = \frac{A_{u\ell}/B_{u\ell}}{(g_\ell B_{\ell u}/g_u B_{u\ell}) \exp\left(h\nu_0/kT \right) - 1}} \quad ⑮$$

Fundamental Equation giving Mean Intensity
in terms of Q.M. transition probabilities & stat. wts.
↓T!

But we know that in Th. Eq. $\overset{\text{Mean Intensity}}{J_\nu} = \overset{\text{Planck function}}{B_\nu} = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}$

and B_ν varies slowly on the scale of virtually any $\Delta\nu$

so $\boxed{\bar{J} = \int_0^\infty \phi(\nu)\, J_\nu\, d\nu = B_\nu}$ ⑯ since $\int_0^\infty \phi(\nu)\, d\nu = 1$

⑩ ↗   ↑ ⑪

So, then for eq ⑮ to give $\bar{J} = B_\nu$, the Einstein coeff are related by:

"detailed balance"

⑰ $\boxed{g_\ell B_{\ell u} = g_u B_{u\ell}}$ $\left\{\begin{array}{l}\text{"absorp \& stim em related} \\ \text{by ratio of stat. wts"}\end{array}\right\}$ $\frac{g_\ell}{g_u} = \frac{B_{u\ell}}{B_{\ell u}}$

⑱ $\boxed{A_{u\ell} = \frac{2h\nu^3}{c^2} B_{u\ell}}$ "amount of spontaneous emission related to stim. emission by $\frac{2h\nu^3}{c^2}$"

note : this is independent of T $\left\{\begin{array}{l}\text{must hold even out of} \\ \text{Th. Eq} - \text{Q.M. property!}\end{array}\right.$

(Prob Set will show why Einstein had to include Stim. Em. to get Planck Law rather than just Wien law $\left(S_\nu = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}\right)$

$h\nu \gg kT$
upper level underpop.

Note: If we know any 1 of $A_{u\ell}, B_{u\ell}, B_{\ell u}$ we can get other two : very USEFUL

Next task: We'll relate this "microscopic, atomic" discussion of atomic/molec states \& rate coeff to "macroscopic" absorption \& emission coeff (More Radiative Xfer, etc)

Highlights of Rybicki & Lightman, Ch. 1...

"3" Introductory Radiative Transfer (Part II) (+ see handout)

Recall $\boxed{I_\nu = \text{constant along rays}}$ (see R&L p. 7)

specific intensity

erg s^{-1} ster^{-1} cm^{-2} Hz^{-1}

$\therefore \boxed{\dfrac{dI_\nu}{ds} = 0}$ ① when there are no extra sources of emission and no absorption & no scattering in/out of beam

S = path length

Ⓠ What changes this and makes $\boxed{\dfrac{dI_\nu}{ds} \neq 0}$?

Ⓐ (i) Emission (ii) Absorption (iii) Scattering

(i) Emission

$\boxed{j = \text{emission coeff} = \dfrac{\text{energy emitted}}{\text{time} \cdot \text{solid} \angle \cdot \text{volume}}}$

$j_\nu = \text{monochromatic emission coeff} = \dfrac{(\text{units of } j)}{\text{freq}}$

So $\boxed{dE = j_\nu \, dV \, d\Omega \, dt \, d\nu}$ ②

more sensible units $\Rightarrow$

$\left[\epsilon_\nu = \text{emissivity} = \dfrac{\text{energy emitted spontaneously}}{\text{frequency} \cdot \text{time} \cdot \text{mass}} = \dfrac{\text{erg}}{\text{gm} \ s \ Hz} \right]$

So $\boxed{dE = \epsilon_\nu \, \rho \, dV \, dt \, d\nu \, \dfrac{d\Omega}{4\pi}}$ ③

mass

$\uparrow$ takes into account the fraction radiated into $d\Omega$

So ② = ③ $\Rightarrow$ $\boxed{j_\nu = \dfrac{\epsilon_\nu \, \rho}{4\pi}}$ ④

And intensity added to beam by spontaneous emission:

$\boxed{dI_\nu = j_\nu \, ds}$ ⑤ $\left(\dfrac{dI_\nu}{dI_\nu} = j_\nu \right)$ for only that

(ii.) Absorption

$$\alpha_\nu (cm^{-1}) = \text{absorption coefficient}$$

Convention: $\alpha_\nu > \emptyset$ for energy <u>removed</u> from beam, so:

Defined by: $$dI_\nu = -\alpha_\nu I_\nu ds \quad \text{⑥}$$ (see R&L p. 10)

↳ has units ¹/s

In terms of the microscopic cross-section, σ, we discussed last time:

$$\alpha_\nu = n\sigma_\nu \quad \text{⑦}$$ where n = #dens of absorbers

K_ν is defined as the "mass absorption coefficient" or the "opacity" and is given by

$$\alpha_\nu = \rho K_\nu \quad \text{⑧}$$ K_ν has units $\frac{cm^2}{g}$

so, obviously ⑦=⑧ ⇒ $\dfrac{\rho}{n} = \dfrac{\sigma_\nu}{K_\nu}$ ⇒ mass per absorber↓ $\mu = \dfrac{\sigma_\nu}{K_\nu}$

or $$K_\nu = \sigma_\nu / \mu \quad \text{⑨}$$

$$\text{opacity} = \left(\frac{\text{x-section}}{\text{mass per absorber}}\right)$$

Note: this K_ν is the opacity for <u>absorption</u> only — don't confuse it with scattering opacity

Note Further {
(Assumed here)
1. $\sigma_\nu^{1/2} \ll d \sim n^{-1/3}$ cross.secn ≪ dist btwn ptcls
2. absorbers independent & randomly distributed (usually OK in ISM)

(ii) continued ... a weird point about α_ν

this "absorption" coefficient INCLUDES stimulated
emission, since it has same αI_ν & just opp. sign
- i.e. $\alpha_\nu < 0$ corresponds to energy <u>added</u> to
beam

α_ν is really $\sum \left[(abs) + (stim \; em) \right]$ & $\sum \lessgtr \emptyset$

(iii) Scattering (we'll return to this later) ...

| "Full" Equation of Radiative Xfer (w/o scattering) |

$$\boxed{\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu} \quad \text{\textcircled{10}}$$
combining ⑤ & ⑥

of course, the trick is finding <u>ALL</u> the relevant contributors
to α_ν & j_ν !! can be <u>many</u> choices

Note:
When scattering is included this gets complicated, because
things are <u>not isotropic</u> w/in the beam anymore.
Often, numerical integration req to include scatt.

To write solution to ⑩

Define Optical Depth, τ_ν

$$\boxed{d\tau_\nu = \alpha_\nu \, ds} \quad \text{⑪a}$$

$$\boxed{\tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s') \, ds'} \quad \text{⑪b}$$

so ⑥ $\Rightarrow$ $\quad$ for just abs
$$dI_\nu = -d\tau_\nu I_\nu$$

ν note this is a fun'n of ν & <u>CAN</u> vary along s