Tech memo 117:
4x4 Mueller matrix relating Stokes visibilities to correlator outputs for arbitrary antenna gain matrices and configurations

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■ Purpose

In view of the common practice of referring to the correlator outputs as "visibilities", the distinction between the correlator outputs and the Stokes Visiblities should not be ignored. The matrix relationship relating the four Stokes Visibilities to Correlator Outputs is important:

(a) as the foundation of AIPS++ in so far as AIPS++ is based on the Hamaker-Bregman-Seault (HBS) direct product formulation for the generation of the matrix (Tim Conway talk at SAO).
(b) in polarimetric-interferometric calibration.
(c) for an organized understanding of the tremendous number of parameters in the SMA uvfits tables (see Zhao's notes and FITS format for VLBA correlators), many of which enter into the determination of the parameters of the matrix

We would like to evaluate the Mueller Matrix given by the HBS direct (or Kronecker) product explicitly, given a model for telescope gain and configuration.

We couldn't find the expression in HBS's papers. These authors, using the error perturbation method given in the standard ellipsometry text of Azzam and Bashara, gave only approximations or special cases of this direct product. They rederived the Black Box formula of Morris, Radhakrishnan and Seielstad (MRS), but left obscure the connection of this equation to their main point: the direct product. We would like:
(a) To give an explicit expression for the overall scattering matrix of antenna pairs, leading to the next step:
(b) To give explicit expressions for the direct product formulation for the 4x4 matrix introduced by Hamaker-Bregman-Seault relating Stokes Visibilities to Correlator Outputs
(c) To show that these expressions reduce to MRS Black Box formula in the special case of isotropic diagonal gain matrices, as a check

Many of the parameters in the UV-FITS tables will even find their way into evaluating one or another component of this Mueller matrix.

For a quick look at this expression, see the Conclusions Section of this memo.

■ Introduction

It seems common to refer to the correlator ouputs as "visibilities", but a distinction must be made between these quantities and the Stokes Visiblities of the incident light. The four incident Stokes Visibilities (Fourier Transforms of Stokes parameters) are related to the four Correlator Outputs (cross-correlations between one pair of detectors from each interferometric arm) by 4x4 matrices called Mueller matrices. Hamaker, Bregman and Sault showed that the matrices are direct products of a pair of Jones matrices for each interferometric arm. The idea is simple, but the task of evaluating the matrices for arbitary parallactic angles, feed configurations and antenna electronic gain matrices is still tedious. We already have an inkling of this in the derivation of the formula of Morris, Radhakrishnan and Seielstad, which HBS calls the Black Box formula. This formula is valid only for isotropic antenna gain matrices (identical gain in all directions). The coefficients of the Stokes Visibilities in this formula, when given four different combinations of parallactic angles and ellipticities, give the four rows of the general Mueller matrix relating the Stokes Visibilities to the Correlator Outputs.

We are motivated to derive this Mueller matrix to appreciate how the Correlator Outputs are related to the Stokes Visibities on a detailed instrument level. Further down the road we will use it to ease the calibation process. The calculation of the elements of the Mueller matrix will draw upon the large number of parameters currently under evaluation for the SMA uvfits format (Zhao), and will be a benchmark on the adequacy of the format.

■ Antenna gain matrices and configuration matrices

Let us consider one interferometer arm first. Let the parallactic angle be ψ. The sky coordinates must be rotated through this angle to coincide with the telescope axes. Let the main gain matrix be diagonal:

```
TraditionalForm[gainMatrix[g11, g22]]
```

$$\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

Following HBS, but adopting a different notation for simplicity in final expressions, let the gain "error" matrix be

```
TraditionalForm[feedErrorMatrix[g12 / g11, g21 / g22]]
```

$$\begin{pmatrix} 1 & \frac{g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{22}} & 1 \end{pmatrix}$$

Then the overall gain matrix is, without loss of generality, the dot product of the main and error gain matrices:

```
TraditionalForm[gainMatrix[g11, g22] . feedErrorMatrix[g12 / g11, g21 / g22]]
```

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

In each interferometer arm, the light has the option of passing through a quarter-wave plate to convert linear to circular polarization (see Scott Paine's waveguide writeup in the Project Book). We will assume, for generality that this quarter-wave plate (birefringent crystal with fast and slow axes) is a special case of a compenator or retardation plate with phase gains $\frac{\chi}{2}$ and lags $-\frac{\chi}{2}$ along the fast and slow axes, and that the fast axis is at angle $-\pi/4 + \delta$ with respect to the telescope x-axis.

```
TraditionalForm[Simplify[retardPlate[χ / 2, -Pi / 4 + δ, 0]]]
```

$$\begin{pmatrix} \cos(\frac{\chi}{4}) + i \sin(2\delta) \sin(\frac{\chi}{4}) & -i \cos(2\delta) \sin(\frac{\chi}{4}) \\ -i \cos(2\delta) \sin(\frac{\chi}{4}) & \cos(\frac{\chi}{4}) - i \sin(2\delta) \sin(\frac{\chi}{4}) \end{pmatrix}$$

If we follow the retardation plate up with an x-polarizer, we get the action:

```
TraditionalForm[Simplify[xPolarizer . retardPlate[χ / 2, -Pi / 4 + δ, 0]]]
```

$$\begin{pmatrix} \cos(\frac{\chi}{4}) + i \sin(2\delta) \sin(\frac{\chi}{4}) & -i \cos(2\delta) \sin(\frac{\chi}{4}) \\ 0 & 0 \end{pmatrix}$$

Notes:

1. δ represents the deviation from the standard 45 degree orientation of the fast/slow axis of the retarder

2.δ -> $\frac{\pi}{2} + \delta$ for the opposite "tack"

■ Examples

The quarter wave plate converts linear to circular polarization.

```
TraditionalForm[Simplify[retardPlate[Pi / 2, -Pi / 4 + δ, 0] ]](*quarter wave plate*)
```

$$\begin{pmatrix} \frac{i \sin(2\delta)+1}{\sqrt{2}} & -\frac{i \cos(2\delta)}{\sqrt{2}} \\ -\frac{i \cos(2\delta)}{\sqrt{2}} & \frac{1-i \sin(2\delta)}{\sqrt{2}} \end{pmatrix}$$

The actions of the half-wave and full-wave plates are:

```
TraditionalForm[retardPlate[Pi, -Pi / 4, 0]] (* Half Wave Plate *)
```

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

```
TraditionalForm[retardPlate[2 * Pi, -Pi / 4, 0]] (* Full Wave Plate *)
```

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The configuration matrix of HBS I is skipped: it is absorbed into the Gain Matrix. The action of the combined parallactic rotation and retardation plate is:

```
TraditionalForm[gainMatrix[g11, g22] . feedErrorMatrix[g12 / g11, g21 / g22] . parallactic[ψ]]
```

$$\begin{pmatrix} \cos(\psi)\, g_{11} - \sin(\psi)\, g_{12} & \sin(\psi)\, g_{11} + \cos(\psi)\, g_{12} \\ \cos(\psi)\, g_{21} - \sin(\psi)\, g_{22} & \sin(\psi)\, g_{21} + \cos(\psi)\, g_{22} \end{pmatrix}$$

```
projection[ϵ, χ, δ] . gainMatrix[g11, g22] . feedErrorMatrix[g12 / g11, g21 / g22] . parallactic[ψ]
```

```
{{Cos[ψ] (E^-I ϵ (Cos[χ] + I Sin[2 δ] Sin[χ]) g11 - I E^-I ϵ Cos[2 δ] Sin[χ] g21) -
    Sin[ψ] (E^-I ϵ (Cos[χ] + I Sin[2 δ] Sin[χ]) g12 - I E^-I ϵ Cos[2 δ] Sin[χ] g22),
  Sin[ψ] (E^-I ϵ (Cos[χ] + I Sin[2 δ] Sin[χ]) g11 - I E^-I ϵ Cos[2 δ] Sin[χ] g21) +
    Cos[ψ] (E^-I ϵ (Cos[χ] + I Sin[2 δ] Sin[χ]) g12 - I E^-I ϵ Cos[2 δ] Sin[χ] g22)},
 {0, 0}}
```

This is the voltage input from a receiver from one antenna to the correlator. Let us the combine this with the voltage input to the correlator from the other receiver, and we will start labeling the different receivers m and n. The gain matrix of the second antenna will be denoted by

```
TraditionalForm[gainMatrix[h11, h22] . feedErrorMatrix[h12 / h11, h21 / h22]]
```

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

```
TraditionalForm[
 Simplify[projection[ϵm, χm, δm] . gainMatrix[g11, g22] . feedErrorMatrix[g12 / g11, g21 / g22] . parallactic[ψm] +
 Reverse[projection[ϵn, χn, δn] . gainMatrix[h11, h22] . feedErrorMatrix[h12 / h11, h21 / h22] . parallactic[ψn]]]]
```

$$\begin{pmatrix} e^{-i\epsilon_m}(\cos(\psi_m)(\cos(\chi_m)+i\sin(2\delta_m)\sin(\chi_m))g_{11}-(\cos(\chi_m)+i\sin(2\delta_m)\sin(\chi_m))\sin(\psi_m)g_{12}-i\cos(2\delta_m)\sin(\chi_m)(\cos(\psi_m)g_{21}-\sin(\psi_m)g_{22})) & e^{-i\epsilon_m}((\cos(\chi_m)+i\sin(2\delta \\ e^{-i\epsilon_n}(\cos(\psi_n)(\cos(\chi_n)+i\sin(2\delta_n)\sin(\chi_n))h_{11}-(\cos(\chi_n)+i\sin(2\delta_n)\sin(\chi_n))\sin(\psi_n)h_{12}-i\cos(2\delta_n)\sin(\chi_n)(\cos(\psi_n)h_{21}-\sin(\psi_n)h_{22})) & e^{-i\epsilon_n}((\cos(\chi_n)+i\sin(2 \end{pmatrix}$$

This may be regarded as an overall "scattering matrix" for two incoming voltage signals. We next apply the convenient formulae in scattering theory (van de Hulst) for converting this scattering matrix into its own Mueller matrix. We need the following results:

Let a general scattering matrix be given by a 2 by 2 matrix

```
TraditionalForm[scatterMatrix]
```

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

Then the Mueller matrix corresponding to this scattering matrix is (van de Hulst):

```
TraditionalForm[muellerMatrix[scatterMatrix]]
```

$$\begin{pmatrix} \frac{1}{2}(|s_{11}|^2+|s_{12}|^2+|s_{21}|^2+|s_{22}|^2) & \frac{1}{2}(|s_{11}|^2-|s_{12}|^2+|s_{21}|^2-|s_{22}|^2) & \text{Re}(\text{Conjugate}(s_{12})\,s_{11}+\text{Conjugate}(s_{22})\,s_{21}) & \text{Im}(\text{Conjugate}(s_{12})\,s_{11}+\text{Conjugate} \\ \frac{1}{2}(|s_{11}|^2+|s_{12}|^2-|s_{21}|^2-|s_{22}|^2) & \frac{1}{2}(|s_{11}|^2-|s_{12}|^2-|s_{21}|^2+|s_{22}|^2) & \text{Re}(\text{Conjugate}(s_{12})\,s_{11}-\text{Conjugate}(s_{22})\,s_{21}) & \text{Im}(\text{Conjugate}(s_{12})\,s_{11}-\text{Conjugate} \\ \text{Re}(\text{Conjugate}(s_{21})\,s_{11}+\text{Conjugate}(s_{22})\,s_{12}) & \text{Re}(\text{Conjugate}(s_{21})\,s_{11}-\text{Conjugate}(s_{22})\,s_{12}) & \text{Re}(\text{Conjugate}(s_{22})\,s_{11}+\text{Conjugate}(s_{21})\,s_{12}) & \text{Im}(\text{Conjugate}(s_{22})\,s_{11}-\text{Conjugate} \\ \text{Im}(\text{Conjugate}(s_{11})\,s_{21}+\text{Conjugate}(s_{12})\,s_{22}) & \text{Im}(\text{Conjugate}(s_{11})\,s_{21}-\text{Conjugate}(s_{12})\,s_{22}) & \text{Im}(\text{Conjugate}(s_{12})\,s_{21}+\text{Conjugate}(s_{11})\,s_{22}) & \text{Re}(\text{Conjugate}(s_{22})\,s_{11}-\text{Conjugate} \end{pmatrix}$$

The Mueller matrix converts an input Stokes vector into an output Stokes vector. What we want is an output coherency matrix. This can be done by premultiplying the Mueller matrix by the matrix:

```
TraditionalForm[Inverse[qMatrix]]
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}$$

to get

```
TraditionalForm[
 Simplify[ComplexExpand[Inverse[qMatrix] .%%, {S11, S12, S21, S22}, TargetFunctions -> {Conjugate, Abs}]]]
```

$$\begin{pmatrix} \frac{1}{2}\left(|s_{11}|^2+|s_{12}|^2\right) & \frac{1}{2}\left(|s_{11}|^2-|s_{12}|^2\right) & \frac{1}{2}\left(\text{Conjugate}(s_{12})\,s_{11}+\text{Conjugate}(s_{11})\,s_{12}\right) & \frac{1}{2}\,i\left(\text{Conjugate}(s_{11})\,s_{12}-\text{Conjugate}\right. \\ \frac{1}{2}\left(\text{Conjugate}(s_{21})\,s_{11}+\text{Conjugate}(s_{22})\,s_{12}\right) & \frac{1}{2}\left(\text{Conjugate}(s_{21})\,s_{11}-\text{Conjugate}(s_{22})\,s_{12}\right) & \frac{1}{2}\left(\text{Conjugate}(s_{22})\,s_{11}+\text{Conjugate}(s_{21})\,s_{12}\right) & \frac{1}{2}\,i\left(\text{Conjugate}(s_{21})\,s_{12}-\text{Conjugate}\right. \\ \frac{1}{2}\left(\text{Conjugate}(s_{11})\,s_{21}+\text{Conjugate}(s_{12})\,s_{22}\right) & \frac{1}{2}\left(\text{Conjugate}(s_{11})\,s_{21}-\text{Conjugate}(s_{12})\,s_{22}\right) & \frac{1}{2}\left(\text{Conjugate}(s_{12})\,s_{21}+\text{Conjugate}(s_{11})\,s_{22}\right) & \frac{1}{2}\,i\left(\text{Conjugate}(s_{11})\,s_{22}-\text{Conjugate}\right. \\ \frac{1}{2}\left(|s_{21}|^2+|s_{22}|^2\right) & \frac{1}{2}\left(|s_{21}|^2-|s_{22}|^2\right) & \frac{1}{2}\left(\text{Conjugate}(s_{22})\,s_{21}+\text{Conjugate}(s_{21})\,s_{22}\right) & \frac{1}{2}\,i\left(\text{Conjugate}(s_{21})\,s_{22}-\text{Conjugate}\right. \end{pmatrix}$$

For convenience I will call this the Morris Matrix. Notice that the first and fourth rows have similar forms, except for a simple permutation of indices. Likewise the second and third rows. Taking the forms of the first and second rows, we can define the cross-correlation and auto-correlation functions.

The cross-correlation function:

```
TraditionalForm[crossCorrelation[scatterMatrix] * Exp[I * (p - q)]]
```

$$\left\{\frac{1}{2}\,e^{i(p-q)}\left(\text{Conjugate}(s_{21})\,s_{11}+\text{Conjugate}(s_{22})\,s_{12}\right),\ \frac{1}{2}\,e^{i(p-q)}\left(\text{Conjugate}(s_{21})\,s_{11}-\text{Conjugate}(s_{22})\,s_{12}\right),\ \frac{1}{2}\,e^{i(p-q)}\left(\text{Conjugate}(s_{22})\,s_{11}+\text{Conjugate}(s_{21})\,s_{12}\right),\right.$$
$$\left.\frac{1}{2}\,i\,e^{i(p-q)}\left(\text{Conjugate}(s_{21})\,s_{12}-\text{Conjugate}(s_{22})\,s_{11}\right)\right\}$$

The auto-correlation function:

```
TraditionalForm[
 Simplify[ComplexExpand[autoCorrelation[Exp[I * p] * {s11, s12}], {s11, s12}, TargetFunctions -> {Abs, Conjugate}]]]
```

$$\left\{\frac{1}{2}\left(|s_{11}|^2+|s_{12}|^2\right),\ \frac{1}{2}\left(|s_{11}|^2-|s_{12}|^2\right),\ \frac{1}{2}\left(\text{Conjugate}(s_{12})\,s_{11}+\text{Conjugate}(s_{11})\,s_{12}\right),\ \frac{1}{2}\,i\left(\text{Conjugate}(s_{11})\,s_{12}-\text{Conjugate}(s_{12})\,s_{11}\right)\right\}$$

■ Auto -correlation between detectors on the same antenna and cross-correlation for detectors from an antenna pair

Let the retardation alignment errors δ be zero (the number of terms increases rapidly if these are included).

```
trans = overallScatter[{0, 0}, {χm, χn}, {0, 0}, {ψm, ψn}, {g11, g12/g11, g21/g22, g22}, {h11, h12/h11, h21/h22, h22}];
```

```
TraditionalForm[trans]
```

$$\begin{pmatrix} \cos(\chi_m)\,g_{11}\left(\cos(\psi_m)-\frac{\sin(\psi_m)\,g_{12}}{g_{11}}\right)-i\sin(\chi_m)\left(\frac{\cos(\psi_m)\,g_{21}}{g_{22}}-\sin(\psi_m)\right)g_{22} & \cos(\chi_m)\,g_{11}\left(\sin(\psi_m)+\frac{\cos(\psi_m)\,g_{12}}{g_{11}}\right)-i\sin(\chi_m)\left(\cos(\psi_m)+\frac{\sin(\psi_m)\,g_{21}}{g_{22}}\right)g_{22} \\ \cos(\chi_n)\,h_{11}\left(\cos(\psi_n)-\frac{\sin(\psi_n)\,h_{12}}{h_{11}}\right)-i\sin(\chi_n)\left(\frac{\cos(\psi_n)\,h_{21}}{h_{22}}-\sin(\psi_n)\right)h_{22} & \cos(\chi_n)\,h_{11}\left(\sin(\psi_n)+\frac{\cos(\psi_n)\,h_{12}}{h_{11}}\right)-i\sin(\chi_n)\left(\cos(\psi_n)+\frac{\sin(\psi_n)\,h_{21}}{h_{22}}\right)h_{22} \end{pmatrix}$$

```
autoCorrelation[trans];
```

```
ComplexExpand[%, {g11, g12, g21, g22}, TargetFunctions -> {Absolute, Conjugate}];
```

TraditionalForm[FullSimplify[%]]

$$\left\{\frac{1}{2}\left(\text{Conjugate}[g_{11}]\,\text{Cos}[\chi_m]^2\,g_{11}\left(1+\frac{\text{Conjugate}[g_{12}]\,g_{12}}{\text{Conjugate}[g_{11}]\,g_{11}}\right)+\text{Conjugate}[g_{22}]\,\text{Sin}[\chi_m]^2\left(1+\frac{\text{Conjugate}[g_{21}]\,g_{21}}{\text{Conjugate}[g_{22}]\,g_{22}}\right)g_{22}-\right.\right.$$
$$\left.\frac{1}{2}\,\text{I}\,\text{Sin}[2\,\chi_m]\left(-\text{Conjugate}[g_{22}]\,g_{11}\left(\frac{\text{Conjugate}[g_{21}]}{\text{Conjugate}[g_{22}]}+\frac{g_{12}}{g_{11}}\right)+\text{Conjugate}[g_{11}]\left(\frac{\text{Conjugate}[g_{12}]}{\text{Conjugate}[g_{11}]}+\frac{g_{21}}{g_{22}}\right)g_{22}\right)\right),$$
$$\frac{1}{4}\left(\text{Cos}[2\,\psi_m]\right.$$
$$\left(-2\,\text{Conjugate}[g_{11}]\,\text{Cos}[\chi_m]^2\,g_{11}\left(-1+\frac{\text{Conjugate}[g_{12}]\,g_{12}}{\text{Conjugate}[g_{11}]\,g_{11}}\right)+2\,\text{Conjugate}[g_{22}]\,\text{Sin}[\chi_m]^2\left(-1+\frac{\text{Conjugate}[g_{21}]\,g_{21}}{\text{Conjugate}[g_{22}]\,g_{22}}\right)g_{22}+\right.$$
$$\left.\text{I}\,\text{Sin}[2\,\chi_m]\left(\text{Conjugate}[g_{22}]\,g_{11}\left(\frac{\text{Conjugate}[g_{21}]}{\text{Conjugate}[g_{22}]}-\frac{g_{12}}{g_{11}}\right)+\text{Conjugate}[g_{11}]\left(\frac{\text{Conjugate}[g_{12}]}{\text{Conjugate}[g_{11}]}-\frac{g_{21}}{g_{22}}\right)g_{22}\right)\right)-$$
$$\text{Sin}[2\,\psi_m]\left(2\,\text{Conjugate}[g_{11}]\,\text{Cos}[\chi_m]^2\,g_{11}\left(\frac{\text{Conjugate}[g_{12}]}{\text{Conjugate}[g_{11}]}+\frac{g_{12}}{g_{11}}\right)+2\,\text{Conjugate}[g_{22}]\,\text{Sin}[\chi_m]^2\left(\frac{\text{Conjugate}[g_{21}]}{\text{Conjugate}[g_{22}]}+\frac{g_{21}}{g_{22}}\right)g_{22}-\right.$$
$$\left.\left.\text{I}\,\text{Sin}[2\,\chi_m]\left(-\text{Conjugate}[g_{22}]\,g_{11}\left(1+\frac{\text{Conjugate}[g_{21}]\,g_{12}}{\text{Conjugate}[g_{22}]\,g_{11}}\right)+\text{Conjugate}[g_{11}]\left(\frac{\text{Conjugate}[g_{12}]\,g_{21}}{\text{Conjugate}[g_{11}]}+g_{22}\right)\right)\right)\right),$$
$$\frac{1}{4}\left(-\text{Sin}[2\,\psi_m]\right.$$
$$\left(2\,\text{Conjugate}[g_{11}]\,\text{Cos}[\chi_m]^2\,g_{11}\left(-1+\frac{\text{Conjugate}[g_{12}]\,g_{12}}{\text{Conjugate}[g_{11}]\,g_{11}}\right)-2\,\text{Conjugate}[g_{22}]\,\text{Sin}[\chi_m]^2\left(-1+\frac{\text{Conjugate}[g_{21}]\,g_{21}}{\text{Conjugate}[g_{22}]\,g_{22}}\right)g_{22}+\right.$$
$$\left.\text{I}\,\text{Sin}[2\,\chi_m]\left(\text{Conjugate}[g_{22}]\,g_{11}\left(-\frac{\text{Conjugate}[g_{21}]}{\text{Conjugate}[g_{22}]}+\frac{g_{12}}{g_{11}}\right)+\text{Conjugate}[g_{11}]\left(-\frac{\text{Conjugate}[g_{12}]}{\text{Conjugate}[g_{11}]}+\frac{g_{21}}{g_{22}}\right)g_{22}\right)\right)+$$
$$\text{Cos}[2\,\psi_m]\left(2\,\text{Conjugate}[g_{11}]\,\text{Cos}[\chi_m]^2\,g_{11}\left(\frac{\text{Conjugate}[g_{12}]}{\text{Conjugate}[g_{11}]}+\frac{g_{12}}{g_{11}}\right)+2\,\text{Conjugate}[g_{22}]\,\text{Sin}[\chi_m]^2\left(\frac{\text{Conjugate}[g_{21}]}{\text{Conjugate}[g_{22}]}+\frac{g_{21}}{g_{22}}\right)g_{22}-\right.$$
$$\left.\left.\text{I}\,\text{Sin}[2\,\chi_m]\left(-\text{Conjugate}[g_{22}]\,g_{11}\left(1+\frac{\text{Conjugate}[g_{21}]\,g_{12}}{\text{Conjugate}[g_{22}]\,g_{11}}\right)+\text{Conjugate}[g_{11}]\left(\frac{\text{Conjugate}[g_{12}]\,g_{21}}{\text{Conjugate}[g_{11}]}+g_{22}\right)\right)\right)\right),$$
$$\frac{1}{4}\left(2\,\text{I}\left(\text{Conjugate}[g_{11}]\,\text{Cos}[\chi_m]^2\,g_{11}\left(-\frac{\text{Conjugate}[g_{12}]}{\text{Conjugate}[g_{11}]}+\frac{g_{12}}{g_{11}}\right)+\text{Conjugate}[g_{22}]\,\text{Sin}[\chi_m]^2\left(\frac{\text{Conjugate}[g_{21}]}{\text{Conjugate}[g_{22}]}-\frac{g_{21}}{g_{22}}\right)g_{22}\right)-\right.$$
$$\left.\text{Sin}[2\,\chi_m]\left(\text{Conjugate}[g_{22}]\,g_{11}\left(-1+\frac{\text{Conjugate}[g_{21}]\,g_{12}}{\text{Conjugate}[g_{22}]\,g_{11}}\right)+\text{Conjugate}[g_{11}]\left(-1+\frac{\text{Conjugate}[g_{12}]\,g_{21}}{\text{Conjugate}[g_{11}]\,g_{22}}\right)g_{22}\right)\right\}$$

crossCorrelation[trans];

ComplexExpand[%, {g_{11}, g_{12}, g_{21}, g_{22}, h_{11}, h_{12}, h_{21}, h_{22}}, TargetFunctions -> {Absolute, Conjugate}];

FullSimplify[4 * %]

$$\Big\{2\cos(\psi_m-\psi_n)\Big(\text{Conjugate}(h_{22})\sin(\chi_n)\Big(i\cos(\chi_m)\,g_{11}\Big(\text{Conjugate(h21)}+\frac{g_{12}}{g_{11}}\Big)+(\text{g21}\,\text{Conjugate(h21)}+1)\sin(\chi_m)\,g_{22}\Big)+$$
$$\text{Conjugate}(h_{11})\cos(\chi_n)\Big(\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate}(h_{12})\,g_{12}}{\text{Conjugate}(h_{11})\,g_{11}}+1\Big)-i\Big(\text{g21}+\frac{\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}\Big)\sin(\chi_m)\,g_{22}\Big)\Big)-$$
$$2\sin(\psi_m-\psi_n)\Big(\text{Conjugate}(h_{22})\sin(\chi_n)\Big(i\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate(h21)}\,g_{12}}{g_{11}}-1\Big)+(\text{Conjugate(h21)}-\text{g21})\sin(\chi_m)\,g_{22}\Big)+$$
$$\text{Conjugate}(h_{11})\cos(\chi_n)\Big(\cos(\chi_m)\,g_{11}\Big(\frac{g_{12}}{g_{11}}-\frac{\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}\Big)+i\Big(\frac{\text{g21}\,\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}-1\Big)\sin(\chi_m)\,g_{22}\Big)\Big),$$
$$\text{Conjugate}(h_{22})\sin(\chi_n)\Big(-2\,i\cos(\chi_m)\,g_{11}\Big(\sin(\psi_m+\psi_n)+\frac{\cos(\psi_m+\psi_n)\,g_{12}}{g_{11}}\Big)-2\sin(\chi_m)\,(\cos(\psi_m+\psi_n)+\text{g21}\sin(\psi_m+\psi_n))\,g_{22}+\text{Conjugate(h21)}$$
$$(i\cos(\chi_m-\psi_m-\psi_n)\,(g_{11}+i\,g_{22})+\cos(\chi_m+\psi_m+\psi_n)\,(i\,g_{11}+g_{22})+\sin(\chi_m+\psi_m+\psi_n)\,(\text{g21}\,g_{22}-i\,g_{12})+\sin(\chi_m-\psi_m-\psi_n)\,(i\,g_{12}+\text{g21}\,g_{22}))\Big)-$$
$$\text{Conjugate}(h_{11})\cos(\chi_n)\Big(2\cos(\psi_m+\psi_n)\Big(\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate}(h_{12})\,g_{12}}{\text{Conjugate}(h_{11})\,g_{11}}-1\Big)+i\Big(\text{g21}-\frac{\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}\Big)\sin(\chi_m)\,g_{22}\Big)+$$
$$2\sin(\psi_m+\psi_n)\Big(\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}+\frac{g_{12}}{g_{11}}\Big)-i\Big(\frac{\text{g21}\,\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}+1\Big)\sin(\chi_m)\,g_{22}\Big)\Big),$$
$$2\Big(\cos(\psi_m+\psi_n)\Big(\text{Conjugate}(h_{22})\sin(\chi_n)\Big(i\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate(h21)}\,g_{12}}{g_{11}}+1\Big)+(\text{g21}+\text{Conjugate(h21)})\sin(\chi_m)\,g_{22}\Big)+$$
$$\text{Conjugate}(h_{11})\cos(\chi_n)\Big(\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}+\frac{g_{12}}{g_{11}}\Big)-i\Big(\frac{\text{g21}\,\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}+1\Big)\sin(\chi_m)\,g_{22}\Big)\Big)-$$
$$\sin(\psi_m+\psi_n)\Big(\text{Conjugate}(h_{22})\sin(\chi_n)\Big(i\cos(\chi_m)\,g_{11}\Big(\frac{g_{12}}{g_{11}}-\text{Conjugate(h21)}\Big)-(\text{g21}\,\text{Conjugate(h21)}-1)\sin(\chi_m)\,g_{22}\Big)+$$
$$\text{Conjugate}(h_{11})\cos(\chi_n)\Big(\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate}(h_{12})\,g_{12}}{\text{Conjugate}(h_{11})\,g_{11}}-1\Big)+i\Big(\text{g21}-\frac{\text{Conjugate}(h_{12})}{\text{Conjugate}(h_{11})}\Big)\sin(\chi_m)\,g_{22}\Big)\Big)\Big),$$
$$2\Big(\text{Conjugate}(h_{22})\sin(\chi_n)\Big(i\sin(\psi_m-\psi_n)\Big(i\cos(\chi_m)\,g_{11}\Big(\text{Conjugate(h21)}+\frac{g_{12}}{g_{11}}\Big)+(\text{g21}\,\text{Conjugate(h21)}+1)\sin(\chi_m)\,g_{22}\Big)-$$
$$\cos(\psi_m-\psi_n)\Big(\cos(\chi_m)\,g_{11}\Big(\frac{\text{Conjugate(h21)}\,g_{12}}{g_{11}}-1\Big)+i\,(\text{g21}-\text{Conjugate(h21)})\sin(\chi_m)\,g_{22}\Big)\Big)+$$
$$\frac{1}{2}\,\text{Conjugate}(h_{11})\cos(\chi_n)\Big(i\sin(\chi_m+\psi_m-\psi_n)\,g_{11}+i\cos(\chi_m-\psi_m+\psi_n)\,g_{12}+$$
$$\text{g21}\cos(\chi_m-\psi_m+\psi_n)\,g_{22}+\sin(\chi_m+\psi_m-\psi_n)\,g_{22}+\sin(\chi_m-\psi_m+\psi_n)\,(g_{22}-i\,g_{11})+\cos(\chi_m+\psi_m-\psi_n)\,(i\,g_{12}-\text{g21}\,g_{22})+$$
$$\frac{1}{\text{Conjugate}(h_{11})}(2\,\text{Conjugate}(h_{12})\,(\sin(\psi_m-\psi_n)\,(i\cos(\chi_m)\,g_{12}+\sin(\chi_m)\,g_{22})-\cos(\psi_m-\psi_n)\,(i\cos(\chi_m)\,g_{11}+\text{g21}\sin(\chi_m)\,g_{22})))\Big)\Big\}$$

We define the autocorrelator and crosscorrelator functions based on these results.

■ Reduction of cross-correlation formula to the Morris et al formula

First consider the autocorrelator formula, which is simpler than the crosscorrelator formula.

```
TraditionalForm[Simplify[autoCorrelator[{{g11, 0}, {0, g22}}, {χ, ψ}]]]
```

$$\left\{\frac{1}{2}\,(\text{Conjugate}(g_{11})\,g_{11}\cos^2(\chi)+\text{Conjugate}(g_{22})\sin^2(\chi)\,g_{22}),\right.$$
$$\frac{1}{4}\,(i\sin(2\chi)\sin(2\psi)\,(\text{Conjugate}(g_{11})\,g_{22}-\text{Conjugate}(g_{22})\,g_{11})+2\cos(2\psi)\,(\text{Conjugate}(g_{11})\cos^2(\chi)\,g_{11}-\text{Conjugate}(g_{22})\sin^2(\chi)\,g_{22})),$$
$$\frac{1}{4}\,(i\cos(2\psi)\sin(2\chi)\,(\text{Conjugate}(g_{22})\,g_{11}-\text{Conjugate}(g_{11})\,g_{22})+2\sin(2\psi)\,(\text{Conjugate}(g_{11})\cos^2(\chi)\,g_{11}-\text{Conjugate}(g_{22})\sin^2(\chi)\,g_{22})),$$
$$\left.\frac{1}{4}\sin(2\chi)\,(\text{Conjugate}(g_{22})\,g_{11}+\text{Conjugate}(g_{11})\,g_{22})\right\}$$

If the gain matrix is isotropic, then this reduces to

```
TraditionalForm[Simplify[autoCorrelator[{{g, 0}, {0, g}}, {χ, ψ}]]]
```

$$\left\{\frac{1}{2}\,g\,\text{Conjugate}(g),\ \frac{1}{2}\,g\,\text{Conjugate}(g)\cos(2\chi)\cos(2\psi),\ \frac{1}{2}\,g\,\text{Conjugate}(g)\cos(2\chi)\sin(2\psi),\ \frac{1}{2}\,g\,\text{Conjugate}(g)\sin(2\chi)\right\}$$

Now we tackle the cross-correlator.

```
crossCorrelator[{{g11, 0}, {0, g22}}, {{h11, 0}, {0, h22}}, {χm, χn}, {ψm, ψn}]
```

```
{1/4 (-2 Sin[ψm - ψn] (-I Conjugate[h22] Cos[χm] Sin[χn] g11 - I Conjugate[h11] Cos[χn] Sin[χm] g22) +
   2 Cos[ψm - ψn] (Conjugate[h11] Cos[χm] Cos[χn] g11 + Conjugate[h22] Sin[χm] Sin[χn] g22)),
 1/4 (Conjugate[h22] Sin[χn] (-2 I Cos[χm] Sin[ψm + ψn] g11 - 2 Cos[ψm + ψn] Sin[χm] g22) -
   Conjugate[h11] Cos[χn] (-2 Cos[χm] Cos[ψm + ψn] g11 - 2 I Sin[χm] Sin[ψm + ψn] g22)),
 1/2 (Cos[ψm + ψn] (I Conjugate[h22] Cos[χm] Sin[χn] g11 - I Conjugate[h11] Cos[χn] Sin[χm] g22) -
   Sin[ψm + ψn] (-Conjugate[h11] Cos[χm] Cos[χn] g11 + Conjugate[h22] Sin[χm] Sin[χn] g22)),
 1/2 (Conjugate[h22] Sin[χn] (Cos[χm] Cos[ψm - ψn] g11 + I Sin[χm] Sin[ψm - ψn] g22) +
   1/2 Conjugate[h11] Cos[χn] (I Sin[χm + ψm - ψn] g11 + Sin[χm + ψm - ψn] g22 + Sin[χm - ψm + ψn] (-I g11 + g22)))}
```

Simplify[%]

$$\left\{\frac{1}{2}\left(\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(\text{I}\,\text{Cos}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{11}+\text{Cos}[\psi_m-\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Cos}[\psi_m-\psi_n]\,g_{11}+\text{I}\,\text{Sin}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{22}\right)\right),\right.$$
$$\frac{1}{2}\left(\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(-\text{I}\,\text{Cos}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{11}-\text{Cos}[\psi_m+\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Cos}[\psi_m+\psi_n]\,g_{11}+\text{I}\,\text{Sin}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{22}\right)\right),$$
$$\frac{1}{2}\left(\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{11}-\text{I}\,\text{Cos}[\psi_m+\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(\text{I}\,\text{Cos}[\chi_m]\,\text{Cos}[\psi_m+\psi_n]\,g_{11}-\text{Sin}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{22}\right)\right),$$
$$\left.\frac{1}{2}\left(\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{I}\,\text{Cos}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{11}+\text{Cos}[\psi_m-\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Cos}[\psi_m-\psi_n]\,g_{11}+\text{I}\,\text{Sin}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{22}\right)\right)\right\}$$

FullSimplify[%]

$$\left\{\frac{1}{2}\left(\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(\text{I}\,\text{Cos}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{11}+\text{Cos}[\psi_m-\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Cos}[\psi_m-\psi_n]\,g_{11}+\text{I}\,\text{Sin}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{22}\right)\right),\right.$$
$$\frac{1}{2}\left(-\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(\text{I}\,\text{Cos}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{11}+\text{Cos}[\psi_m+\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Cos}[\psi_m+\psi_n]\,g_{11}+\text{I}\,\text{Sin}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{22}\right)\right),$$
$$\frac{1}{2}\left(\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{11}-\text{I}\,\text{Cos}[\psi_m+\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(\text{I}\,\text{Cos}[\chi_m]\,\text{Cos}[\psi_m+\psi_n]\,g_{11}-\text{Sin}[\chi_m]\,\text{Sin}[\psi_m+\psi_n]\,g_{22}\right)\right),$$
$$\left.\frac{1}{2}\left(\text{Conjugate}[h_{11}]\,\text{Cos}[\chi_n]\left(\text{I}\,\text{Cos}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{11}+\text{Cos}[\psi_m-\psi_n]\,\text{Sin}[\chi_m]\,g_{22}\right)+\text{Conjugate}[h_{22}]\,\text{Sin}[\chi_n]\left(\text{Cos}[\chi_m]\,\text{Cos}[\psi_m-\psi_n]\,g_{11}+\text{I}\,\text{Sin}[\chi_m]\,\text{Sin}[\psi_m-\psi_n]\,g_{22}\right)\right)\right\}$$

When the antenna gains are istropic but not necessarily equal, $h_{22} \to h_{11}$, $g_{22} \to g_{11}$, and we get

$$\left\{\frac{1}{4}\,\text{Conjugate}[h_{11}]\left(\text{Cos}[\chi_m-\chi_n+\psi_m-\psi_n]-\text{I}\,\text{Cos}[\chi_m+\chi_n+\psi_m-\psi_n]+\text{Cos}[\chi_m-\chi_n-\psi_m+\psi_n]+\text{I}\,\text{Cos}[\chi_m+\chi_n-\psi_m+\psi_n]\right)g_{11},\right.$$
$$\frac{1}{4}\,\text{Conjugate}[h_{11}]\left(\text{I}\,\text{Cos}[\chi_m-\chi_n-\psi_m-\psi_n]+\text{Cos}[\chi_m+\chi_n-\psi_m-\psi_n]-\text{I}\,\text{Cos}[\chi_m-\chi_n+\psi_m+\psi_n]+\text{Cos}[\chi_m+\chi_n+\psi_m+\psi_n]\right)g_{11},$$
$$\frac{1}{2}\,\text{Conjugate}[h_{11}]\left(-\text{I}\,\text{Cos}[\psi_m+\psi_n]\,\text{Sin}[\chi_m-\chi_n]+\text{Cos}[\chi_m+\chi_n]\,\text{Sin}[\psi_m+\psi_n]\right)g_{11},$$
$$\left.\frac{1}{4}\,\text{Conjugate}[h_{11}]\left(\text{I}\,\text{Sin}[\chi_m-\chi_n+\psi_m-\psi_n]+\text{Sin}[\chi_m+\chi_n+\psi_m-\psi_n]-\text{I}\,\text{Sin}[\chi_m-\chi_n-\psi_m+\psi_n]+\text{Sin}[\chi_m+\chi_n-\psi_m+\psi_n]\right)g_{11}\right\}$$

Using trig identities for sums and differences of arguments, and dotting into the Stokes Visibilityh $\{s_0,s_1,s_2,s_3\}$(we avoid using U and V to avoid confusion with the U nd V axes) we can see that this is the Black Box formula of MRS formula.

■ Auto-correlation a special case of cross-correlation

We verify that auto-correlation is actually a special case of cross-correlation

```
TraditionalForm[
 Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{g11, g12 / g11}, {g21 / g22, g22}}, {χ, χ}, {ψ, ψ}]]]
```

$$\left\{\frac{1}{2}\left(\text{Conjugate}(g_{11})\cos(\chi)\left(\cos(\chi)\,g_{11}-i\sin(\chi)\,g_{21}\right)+\right.\right.$$
$$\text{Conjugate}(g_{21})\sin(\chi)\left(i\cos(\chi)\,g_{11}+\sin(\chi)\,g_{21}\right)+\left(\text{Conjugate}(g_{12})\cos(\chi)+i\,\text{Conjugate}(g_{22})\sin(\chi)\right)\left(\cos(\chi)\,g_{12}-i\sin(\chi)\,g_{22}\right)),$$
$$\frac{1}{4}\left(\text{Conjugate}(g_{21})\sin(\chi)\left(i\cos(\chi-2\psi)\,g_{11}+i\cos(\chi+2\psi)\,g_{11}+i\sin(\chi-2\psi)\,g_{12}-\right.\right.$$
$$i\sin(\chi+2\psi)\,g_{12}+\sin(\chi-2\psi)\,g_{21}+\sin(\chi+2\psi)\,g_{21}-\cos(\chi-2\psi)\,g_{22}+\cos(\chi+2\psi)\,g_{22}\Big)+2\,\text{Conjugate}(g_{11})\cos(\chi)$$
$$\left(\cos(\chi)\cos(2\psi)\,g_{11}-\cos(\chi)\sin(2\psi)\,g_{12}-i\cos(2\psi)\sin(\chi)\,g_{21}+i\sin(\chi)\sin(2\psi)\,g_{22}\right)-\left(\text{Conjugate}(g_{12})\cos(\chi)+i\,\text{Conjugate}(g_{22})\sin(\chi)\right)$$
$$\left.\left(-\sin(\chi-2\psi)\,g_{11}+\sin(\chi+2\psi)\,g_{11}+\cos(\chi-2\psi)\,g_{12}+\cos(\chi+2\psi)\,g_{12}-i\cos(\chi-2\psi)\,g_{21}+i\cos(\chi+2\psi)\,g_{21}-i\sin(\chi-2\psi)\,g_{22}-i\sin(\chi+2\psi)\,g_{22}\right)\right),$$
$$\frac{1}{2}\left(\left(\text{Conjugate}(g_{11})\cos(\chi)+i\,\text{Conjugate}(g_{21})\sin(\chi)\right)\left(\cos(\chi)\sin(2\psi)\,g_{11}+\cos(\chi)\cos(2\psi)\,g_{12}-i\sin(\chi)\sin(2\psi)\,g_{21}-i\cos(2\psi)\sin(\chi)\,g_{22}\right)+\right.$$
$$\text{Conjugate}(g_{22})\sin(\chi)\left(i\cos(\chi)\cos(2\psi)\,g_{11}-i\cos(\chi)\sin(2\psi)\,g_{12}+\cos(2\psi)\sin(\chi)\,g_{21}-\sin(\chi)\sin(2\psi)\,g_{22}\right)+$$
$$\left.\text{Conjugate}(g_{12})\cos(\chi)\left(\cos(\chi)\cos(2\psi)\,g_{11}-\cos(\chi)\sin(2\psi)\,g_{12}-i\cos(2\psi)\sin(\chi)\,g_{21}+i\sin(\chi)\sin(2\psi)\,g_{22}\right)\right),$$
$$\frac{1}{2}\left(-i\,\text{Conjugate}(g_{12})\cos(\chi)\left(\cos(\chi)\,g_{11}-i\sin(\chi)\,g_{21}\right)+\text{Conjugate}(g_{22})\sin(\chi)\left(\cos(\chi)\,g_{11}-i\sin(\chi)\,g_{21}\right)+\right.$$
$$\left.\left.\left(\text{Conjugate}(g_{11})\cos(\chi)+i\,\text{Conjugate}(g_{21})\sin(\chi)\right)\left(i\cos(\chi)\,g_{12}+\sin(\chi)\,g_{22}\right)\right)\right\}$$

Special Cases

■ Parallel feeds

```
TraditionalForm[
 Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {χ, χ}, {ψ, ψ}]]]
```

$$\Big\{\frac{1}{2}\,(\text{Conjugate(h11)}\cos(\chi)\,(\text{g11}\cos(\chi)-i\,\text{g21}\sin(\chi))+\text{Conjugate(h12)}\cos(\chi)\,(\text{g12}\cos(\chi)-i\,\text{g22}\sin(\chi))+$$
$$\sin(\chi)\,(i\,\text{g11}\,\text{Conjugate(h21)}\cos(\chi)+i\,\text{g12}\,\text{Conjugate(h22)}\cos(\chi)+\text{g21}\,\text{Conjugate(h21)}\sin(\chi)+\text{g22}\,\text{Conjugate(h22)}\sin(\chi))),$$
$$\frac{1}{4}\Big(2\cos(\chi)\,(\text{Conjugate(h11)}\,(\cos(\chi)\,(\text{g11}\cos(2\psi)-\text{g12}\sin(2\psi))-i\sin(\chi)\,(\text{g21}\cos(2\psi)-\text{g22}\sin(2\psi)))-$$
$$\text{Conjugate(h12)}\,(\text{g12}\cos(\chi)\cos(2\psi)-i\,\text{g22}\sin(\chi)\cos(2\psi)+\text{g11}\cos(\chi)\sin(2\psi)-i\,\text{g21}\sin(\chi)\sin(2\psi)))+$$
$$\text{Conjugate(h22)}\sin(\chi)\Big(-2\,i\cos(\chi)\,(\text{g12}\cos(2\psi)+\text{g11}\sin(2\psi))-2\sin(\chi)\,(\text{g22}\cos(2\psi)+\text{g21}\sin(2\psi))+$$
$$\frac{1}{\text{Conjugate(h22)}}\,(\text{Conjugate(h21)}\,(i\,(\text{g11}+i\,\text{g22})\cos(\chi-2\psi)+(i\,\text{g11}+\text{g22})\cos(\chi+2\psi)+(i\,\text{g12}+\text{g21})\sin(\chi-2\psi)+(\text{g21}-i\,\text{g12})\sin(\chi+2\psi)))\Big)\Big),$$
$$\frac{1}{2}\,(\text{Conjugate(h11)}\cos(\chi)\,(\text{g12}\cos(\chi)\cos(2\psi)-i\,\text{g22}\sin(\chi)\cos(2\psi)+\text{g11}\cos(\chi)\sin(2\psi)-i\,\text{g21}\sin(\chi)\sin(2\psi))+\sin(\chi)$$
$$(i\,\text{g12}\,\text{Conjugate(h21)}\cos(\chi)\cos(2\psi)+i\,\text{g11}\,\text{Conjugate(h22)}\cos(\chi)\cos(2\psi)+\text{g22}\,\text{Conjugate(h21)}\sin(\chi)\cos(2\psi)+\text{g21}\,\text{Conjugate(h22)}\sin(\chi)\cos(2\psi)+$$
$$i\,\text{g11}\,\text{Conjugate(h21)}\cos(\chi)\sin(2\psi)-i\,\text{g12}\,\text{Conjugate(h22)}\cos(\chi)\sin(2\psi)+\text{g21}\,\text{Conjugate(h21)}\sin(\chi)\sin(2\psi)-\text{g22}\,\text{Conjugate(h22)}\sin(\chi)\sin(2\psi))+$$
$$\text{Conjugate(h12)}\cos(\chi)\,(\cos(\chi)\,(\text{g11}\cos(2\psi)-\text{g12}\sin(2\psi))-i\sin(\chi)\,(\text{g21}\cos(2\psi)-\text{g22}\sin(2\psi)))),$$
$$\frac{1}{2}\,(-i\,\text{Conjugate(h12)}\cos(\chi)\,(\text{g11}\cos(\chi)-i\,\text{g21}\sin(\chi))+\text{Conjugate(h11)}\cos(\chi)\,(i\,\text{g12}\cos(\chi)+\text{g22}\sin(\chi))+$$
$$\sin(\chi)\,(-\text{g12}\,\text{Conjugate(h21)}\cos(\chi)+\text{g11}\,\text{Conjugate(h22)}\cos(\chi)+i\,\text{g22}\,\text{Conjugate(h21)}\sin(\chi)-i\,\text{g21}\,\text{Conjugate(h22)}\sin(\chi)))\Big\}$$

■ Parallel linear feeds

```
TraditionalForm[
 Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {ψ, ψ}]]]
```

$$\Big\{\frac{1}{2}\,(\text{g11}\,\text{Conjugate(h11)}+\text{g12}\,\text{Conjugate(h12)}),\ \frac{1}{2}\,(\text{Conjugate(h11)}\,(\text{g11}\cos(2\psi)-\text{g12}\sin(2\psi))-\text{Conjugate(h12)}\,(\text{g12}\cos(2\psi)+\text{g11}\sin(2\psi))),$$
$$\frac{1}{2}\,(\text{Conjugate(h11)}\,(\text{g12}\cos(2\psi)+\text{g11}\sin(2\psi))+\text{Conjugate(h12)}\,(\text{g11}\cos(2\psi)-\text{g12}\sin(2\psi))),\ \frac{1}{2}\,i\,(\text{g12}\,\text{Conjugate(h11)}-\text{g11}\,\text{Conjugate(h12)})\Big\}$$

■ Oppositely polarized feeds

```
TraditionalForm[
 Simplify[crossCorrelator[{{g11, g12}, {g21, g22}}, {{h11, h12}, {h21, h22}}, {χ, -χ}, {ψ, ψ + Pi / 2}]]]
```

$$\Big\{\frac{1}{2}\,(\text{Conjugate(h11)}\cos(\chi)\,(\text{g11}\,(\text{g12}-\text{Conjugate(h12)})\cos(\chi)+i\,\text{g22}\,(\text{g21}\,\text{Conjugate(h12)}-1)\sin(\chi))-$$
$$\text{Conjugate(h22)}\sin(\chi)\,(i\,\text{g11}\,(\text{g12}\,\text{Conjugate(h21)}-1)\cos(\chi)+\text{g22}\,(\text{Conjugate(h21)}-\text{g21})\sin(\chi))),$$
$$\frac{1}{4}\,(-\text{Conjugate(h11)}\cos(\chi)\,(2\cos(2\psi)\,(\text{g11}\,(\text{g12}+\text{Conjugate(h12)})\cos(\chi)-i\,\text{g22}\,(\text{g21}\,\text{Conjugate(h12)}+1)\sin(\chi))-$$
$$2\,(\text{g11}\,(\text{g12}\,\text{Conjugate(h12)}-1)\cos(\chi)+i\,\text{g22}\,(\text{g21}-\text{Conjugate(h12)})\sin(\chi))\sin(2\psi))-$$
$$\text{Conjugate(h22)}\sin(\chi)\,(-2\,\text{g22}\sin(\chi)\,(\text{g21}\cos(2\psi)-\sin(2\psi))-2\,i\,\text{g11}\cos(\chi)\,(\cos(2\psi)-\text{g12}\sin(2\psi))+$$
$$\text{Conjugate(h21)}\,(-(i\,\text{g11}\,\text{g12}+\text{g21}\,\text{g22})\cos(\chi-2\psi)+(\text{g21}\,\text{g22}-i\,\text{g11}\,\text{g12})\cos(\chi+2\psi)+i\,(\text{g11}+i\,\text{g22})\sin(\chi-2\psi)-(i\,\text{g11}+\text{g22})\sin(\chi+2\psi))),$$
$$\frac{1}{2}\,(-\cos(2\psi)\,(\text{Conjugate(h11)}\cos(\chi)\,(\text{g11}\,(\text{g12}\,\text{Conjugate(h12)}-1)\cos(\chi)+i\,\text{g22}\,(\text{g21}-\text{Conjugate(h12)})\sin(\chi))-$$
$$\text{Conjugate(h22)}\sin(\chi)\,(i\,\text{g11}\,(\text{g12}-\text{Conjugate(h21)})\cos(\chi)-\text{g22}\,(\text{g21}\,\text{Conjugate(h21)}-1)\sin(\chi)))-$$
$$(\text{Conjugate(h11)}\cos(\chi)\,(\text{g11}\,(\text{g12}+\text{Conjugate(h12)})\cos(\chi)-i\,\text{g22}\,(\text{g21}\,\text{Conjugate(h12)}+1)\sin(\chi))-$$
$$\text{Conjugate(h22)}\sin(\chi)\,(i\,\text{g11}\,(\text{g12}\,\text{Conjugate(h21)}+1)\cos(\chi)+\text{g22}\,(\text{g21}+\text{Conjugate(h21)})\sin(\chi)))\sin(2\psi)),$$
$$\frac{1}{2}\,(i\,\text{Conjugate(h22)}\sin(\chi)\,(i\,\text{g11}\,\text{g12}\cos(\chi)+\text{g22}\sin(\chi)+\text{Conjugate(h21)}\,(i\,\text{g11}\cos(\chi)+\text{g21}\,\text{g22}\sin(\chi)))-$$
$$i\,\text{Conjugate(h11)}\cos(\chi)\,(\text{g11}\cos(\chi)-i\,\text{g21}\,\text{g22}\sin(\chi)+\text{Conjugate(h12)}\,(\text{g11}\,\text{g12}\cos(\chi)-i\,\text{g22}\sin(\chi)))\Big\}$$

We now work out some special cases that are mentioned in the Thompson-Moran-Swenson textbook.

■ Linear polarization

In each of the following positions, the four vectors together form the rows of the 4x4 Mueller matrix taking Stokes Visibilities to correlator outputs.

■ Position I

```
TraditionalForm[Simplify[
  crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {ψ, ψ + Pi / 2}]]]
```

$$\left\{\frac{1}{2}\,(\text{g12 Conjugate(h11)} - \text{g11 Conjugate(h12)}),\ \frac{1}{2}\,(\text{Conjugate(h12)}\,(\text{g12}\sin(2\,\psi) - \text{g11}\cos(2\,\psi)) - \text{Conjugate(h11)}\,(\text{g12}\cos(2\,\psi) + \text{g11}\sin(2\,\psi))),\right.$$
$$\left.\frac{1}{2}\,(\text{Conjugate(h11)}\,(\text{g11}\cos(2\,\psi) - \text{g12}\sin(2\,\psi)) - \text{Conjugate(h12)}\,(\text{g12}\cos(2\,\psi) + \text{g11}\sin(2\,\psi))),\ -\frac{1}{2}\,i\,(\text{g11 Conjugate(h11)} + \text{g12 Conjugate(h12)})\right\}$$

```
TraditionalForm[
 Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {0, 0}]]]
```

$$\left\{\frac{1}{2}\,(\text{g11 Conjugate(h11)} + \text{g12 Conjugate(h12)}),\ \frac{1}{2}\,(\text{g11 Conjugate(h11)} - \text{g12 Conjugate(h12)}),\ \frac{1}{2}\,(\text{g12 Conjugate(h11)} + \text{g11 Conjugate(h12)}),\right.$$
$$\left.\frac{1}{2}\,i\,(\text{g12 Conjugate(h11)} - \text{g11 Conjugate(h12)})\right\}$$

```
TraditionalForm[Simplify[
  crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {Pi / 2, Pi / 2}]]]
```

$$\left\{\frac{1}{2}\,(\text{g11 Conjugate(h11)} + \text{g12 Conjugate(h12)}),\ \frac{1}{2}\,(\text{g12 Conjugate(h12)} - \text{g11 Conjugate(h11)}),\ \frac{1}{2}\,(-\text{g12 Conjugate(h11)} - \text{g11 Conjugate(h12)}),\right.$$
$$\left.\frac{1}{2}\,i\,(\text{g12 Conjugate(h11)} - \text{g11 Conjugate(h12)})\right\}$$

```
TraditionalForm[
 Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {0, Pi / 2}]]]
```

$$\left\{\frac{1}{2}\,(\text{g12 Conjugate(h11)} - \text{g11 Conjugate(h12)}),\ \frac{1}{2}\,(-\text{g12 Conjugate(h11)} - \text{g11 Conjugate(h12)}),\ \frac{1}{2}\,(\text{g11 Conjugate(h11)} - \text{g12 Conjugate(h12)}),\right.$$
$$\left.-\frac{1}{2}\,i\,(\text{g11 Conjugate(h11)} + \text{g12 Conjugate(h12)})\right\}$$

```
TraditionalForm[
 Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {Pi / 2, 0}]]]
```

$$\left\{\frac{1}{2}\,(\text{g11 Conjugate(h12)} - \text{g12 Conjugate(h11)}),\ \frac{1}{2}\,(-\text{g12 Conjugate(h11)} - \text{g11 Conjugate(h12)}),\ \frac{1}{2}\,(\text{g11 Conjugate(h11)} - \text{g12 Conjugate(h12)}),\right.$$
$$\left.\frac{1}{2}\,i\,(\text{g11 Conjugate(h11)} + \text{g12 Conjugate(h12)})\right\}$$

■ Position II

```
TraditionalForm[Simplify[
  crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {Pi / 4, Pi / 4}]]]
```

$$\left\{\frac{1}{2}(\text{g11 Conjugate(h11)}+\text{g12 Conjugate(h12)}),\ \frac{1}{2}(-\text{g12 Conjugate(h11)}-\text{g11 Conjugate(h12)}),\ \frac{1}{2}(\text{g11 Conjugate(h11)}-\text{g12 Conjugate(h12)}),\ \frac{1}{2}i(\text{g12 Conjugate(h11)}-\text{g11 Conjugate(h12)})\right\}$$

```
TraditionalForm[Simplify[
  crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {3 Pi / 4, 3 Pi / 4}]]]
```

$$\left\{\frac{1}{2}(\text{g11 Conjugate(h11)}+\text{g12 Conjugate(h12)}),\ \frac{1}{2}(\text{g12 Conjugate(h11)}+\text{g11 Conjugate(h12)}),\ \frac{1}{2}(\text{g12 Conjugate(h12)}-\text{g11 Conjugate(h11)}),\ \frac{1}{2}i(\text{g12 Conjugate(h11)}-\text{g11 Conjugate(h12)})\right\}$$

```
TraditionalForm[Simplify[
  crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {Pi / 4, 3 Pi / 4}]]]
```

$$\left\{\frac{1}{2}(\text{g12 Conjugate(h11)}-\text{g11 Conjugate(h12)}),\ \frac{1}{2}(\text{g12 Conjugate(h12)}-\text{g11 Conjugate(h11)}),\ \frac{1}{2}(-\text{g12 Conjugate(h11)}-\text{g11 Conjugate(h12)}),\ -\frac{1}{2}i(\text{g11 Conjugate(h11)}+\text{g12 Conjugate(h12)})\right\}$$

```
TraditionalForm[Simplify[
  crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {0, 0}, {3 Pi / 4, Pi / 4}]]]
```

$$\left\{\frac{1}{2}(\text{g11 Conjugate(h12)}-\text{g12 Conjugate(h11)}),\ \frac{1}{2}(\text{g12 Conjugate(h12)}-\text{g11 Conjugate(h11)}),\ \frac{1}{2}(-\text{g12 Conjugate(h11)}-\text{g11 Conjugate(h12)}),\ \frac{1}{2}i(\text{g11 Conjugate(h11)}+\text{g12 Conjugate(h12)})\right\}$$

■ Quarterwave plate: circular polarization

```
TraditionalForm[Simplify[
  crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {-Pi / 4, -Pi / 4}, {0, 0}]]]
```

$$\left\{\frac{1}{4}\left((g11 + i\, g21)\,\text{Conjugate}(h11) + (g12 + i\, g22)\,\text{Conjugate}(h12) - i\, g11\,\text{Conjugate}(h21) + g21\,\text{Conjugate}(h21) - i\, g12\,\text{Conjugate}(h22) + g22\,\text{Conjugate}(h22)\right),\right.$$
$$\frac{1}{4}\left((g11 + i\, g21)\,\text{Conjugate}(h11) - (g12 + i\, g22)\,\text{Conjugate}(h12) - i\, g11\,\text{Conjugate}(h21) + g21\,\text{Conjugate}(h21) + i\, g12\,\text{Conjugate}(h22) - g22\,\text{Conjugate}(h22)\right),$$
$$\frac{1}{4}\left((g12 + i\, g22)\,\text{Conjugate}(h11) + (g11 + i\, g21)\,\text{Conjugate}(h12) - i\, g12\,\text{Conjugate}(h21) + g22\,\text{Conjugate}(h21) - i\, g11\,\text{Conjugate}(h22) + g21\,\text{Conjugate}(h22)\right),$$
$$\left.\frac{1}{4}\left(i\,(g12 + i\, g22)\,\text{Conjugate}(h11) + (g21 - i\, g11)\,\text{Conjugate}(h12) + g12\,\text{Conjugate}(h21) + i\, g22\,\text{Conjugate}(h21) - g11\,\text{Conjugate}(h22) - i\, g21\,\text{Conjugate}(h22)\right)\right\}$$

```
TraditionalForm[Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}},
  {-Pi / 4, -Pi / 4}, {Pi / 2, 0}]]]
```

$$\left\{\frac{1}{4}\left(-(g12 + i\, g22)\,\text{Conjugate}(h11) + (g11 + i\, g21)\,\text{Conjugate}(h12) + i\, g12\,\text{Conjugate}(h21) - g22\,\text{Conjugate}(h21) - i\, g11\,\text{Conjugate}(h22) + g21\,\text{Conjugate}(h22)\right),\right.$$
$$\frac{1}{4}\left(-g12\,\text{Conjugate}(h11) - i\, g22\,\text{Conjugate}(h11) - g11\,\text{Conjugate}(h12) - i\, g21\,\text{Conjugate}(h12) + i\, g12\,\text{Conjugate}(h21) - g22\,\text{Conjugate}(h21) + i\, g11\,\text{Conjugate}(h22) - g21\,\text{Conjugate}(h22)\right),$$
$$\frac{1}{4}\left((g11 + i\, g21)\,\text{Conjugate}(h11) - (g12 + i\, g22)\,\text{Conjugate}(h12) - i\, g11\,\text{Conjugate}(h21) + g21\,\text{Conjugate}(h21) + i\, g12\,\text{Conjugate}(h22) - g22\,\text{Conjugate}(h22)\right),$$
$$\left.\frac{1}{4}\, i\left((g11 + i\, g21)\,\text{Conjugate}(h11) + (g12 + i\, g22)\,\text{Conjugate}(h12) - i\, g11\,\text{Conjugate}(h21) + g21\,\text{Conjugate}(h21) - i\, g12\,\text{Conjugate}(h22) + g22\,\text{Conjugate}(h22)\right)\right\}$$

```
TraditionalForm[Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}},
  {-Pi / 4, -Pi / 4}, {0, Pi / 2}]]]
```

$$\left\{\frac{1}{4}\left((g12 + i\, g22)\,\text{Conjugate}(h11) - (g11 + i\, g21)\,\text{Conjugate}(h12) - i\, g12\,\text{Conjugate}(h21) + g22\,\text{Conjugate}(h21) + i\, g11\,\text{Conjugate}(h22) - g21\,\text{Conjugate}(h22)\right),\right.$$
$$\frac{1}{4}\left(-g12\,\text{Conjugate}(h11) - i\, g22\,\text{Conjugate}(h11) - g11\,\text{Conjugate}(h12) - i\, g21\,\text{Conjugate}(h12) + i\, g12\,\text{Conjugate}(h21) - g22\,\text{Conjugate}(h21) + i\, g11\,\text{Conjugate}(h22) - g21\,\text{Conjugate}(h22)\right),$$
$$\frac{1}{4}\left((g11 + i\, g21)\,\text{Conjugate}(h11) - (g12 + i\, g22)\,\text{Conjugate}(h12) - i\, g11\,\text{Conjugate}(h21) + g21\,\text{Conjugate}(h21) + i\, g12\,\text{Conjugate}(h22) - g22\,\text{Conjugate}(h22)\right),$$
$$\left.-\frac{1}{4}\, i\left((g11 + i\, g21)\,\text{Conjugate}(h11) + (g12 + i\, g22)\,\text{Conjugate}(h12) - i\, g11\,\text{Conjugate}(h21) + g21\,\text{Conjugate}(h21) - i\, g12\,\text{Conjugate}(h22) + g22\,\text{Conjugate}(h22)\right)\right\}$$

```
TraditionalForm[Simplify[crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}},
   {-Pi / 4, -Pi / 4}, {Pi / 2, Pi / 2}]]]
```

$$\left\{\frac{1}{4}\left((g11 + i\,g21)\,\text{Conjugate}(h11) + (g12 + i\,g22)\,\text{Conjugate}(h12) - i\,g11\,\text{Conjugate}(h21) + g21\,\text{Conjugate}(h21) - i\,g12\,\text{Conjugate}(h22) + g22\,\text{Conjugate}(h22)\right),\right.$$
$$\frac{1}{4}\left(-(g11 + i\,g21)\,\text{Conjugate}(h11) + (g12 + i\,g22)\,\text{Conjugate}(h12) + i\,g11\,\text{Conjugate}(h21) - g21\,\text{Conjugate}(h21) - i\,g12\,\text{Conjugate}(h22) + g22\,\text{Conjugate}(h22)\right),$$
$$\frac{1}{4}\left(-g12\,\text{Conjugate}(h11) - i\,g22\,\text{Conjugate}(h11) -\right.$$
$$\left.g11\,\text{Conjugate}(h12) - i\,g21\,\text{Conjugate}(h12) + i\,g12\,\text{Conjugate}(h21) - g22\,\text{Conjugate}(h21) + i\,g11\,\text{Conjugate}(h22) - g21\,\text{Conjugate}(h22)\right),$$
$$\left.\frac{1}{4}\left(i\,(g12 + i\,g22)\,\text{Conjugate}(h11) + (g21 - i\,g11)\,\text{Conjugate}(h12) + g12\,\text{Conjugate}(h21) + i\,g22\,\text{Conjugate}(h21) - g11\,\text{Conjugate}(h22) - i\,g21\,\text{Conjugate}(h22)\right)\right\}$$

Like the linear polarization cases before, these form the rows of a 4x4 Mueller matrix taking the Stokes Visibilities to correlator outputs.

■ Conclusions

We have derived the generalization of the MRS formula that takes Stokes Visibilities to correlator output for general antenna gain matrices, ellipticities and parallactic angles:

```
TraditionalForm[
 crossCorrelator[{{g11, g12 / g11}, {g21 / g22, g22}}, {{h11, h12 / h11}, {h21 / h22, h22}}, {χm, χn}, {ψm, ψn}] .
  {S0, S1, S2, S3}]
```

$$\frac{1}{4}\Big(2\cos(\psi_m-\psi_n)\Big(\text{Conjugate(h11)}\cos(\chi_n)\Big(\text{g11}\Big(\frac{\text{g12 Conjugate(h12)}}{\text{g11 Conjugate(h11)}}+1\Big)\cos(\chi_m)-i\,\text{g22}\Big(\frac{\text{g21}}{\text{g22}}+\frac{\text{Conjugate(h12)}}{\text{Conjugate(h11)}}\Big)\sin(\chi_m)\Big)+$$
$$\text{Conjugate(h22)}\Big(i\,\text{g11}\Big(\frac{\text{g12}}{\text{g11}}+\frac{\text{Conjugate(h21)}}{\text{Conjugate(h22)}}\Big)\cos(\chi_m)+\text{g22}\Big(\frac{\text{g21 Conjugate(h21)}}{\text{g22 Conjugate(h22)}}+1\Big)\sin(\chi_m)\Big)\sin(\chi_n)\Big)-$$
$$2\Big(\text{Conjugate(h11)}\cos(\chi_n)\Big(\text{g11}\Big(\frac{\text{g12}}{\text{g11}}-\frac{\text{Conjugate(h12)}}{\text{Conjugate(h11)}}\Big)\cos(\chi_m)+i\,\text{g22}\Big(\frac{\text{g21 Conjugate(h12)}}{\text{g22 Conjugate(h11)}}-1\Big)\sin(\chi_m)\Big)+$$
$$\text{Conjugate(h22)}\Big(i\,\text{g11}\Big(\frac{\text{g12 Conjugate(h21)}}{\text{g11 Conjugate(h22)}}-1\Big)\cos(\chi_m)+\text{g22}\Big(\frac{\text{Conjugate(h21)}}{\text{Conjugate(h22)}}-\frac{\text{g21}}{\text{g22}}\Big)\sin(\chi_m)\Big)\sin(\chi_n)\Big)\sin(\psi_m-\psi_n)\Big)$$

s_0 +

$$\frac{1}{4}\Big(\text{Conjugate(h22)}\sin(\chi_n)$$
$$\Big(-2\,i\,\text{g11}\cos(\chi_m)\Big(\frac{\text{g12}\cos(\psi_m+\psi_n)}{\text{g11}}+\sin(\psi_m+\psi_n)\Big)-2\,\text{g22}\sin(\chi_m)\Big(\cos(\psi_m+\psi_n)+\frac{\text{g21}\sin(\psi_m+\psi_n)}{\text{g22}}\Big)+\frac{1}{\text{Conjugate(h22)}}(\text{Conjugate(h21)}$$
$$(i\,(\text{g11}+i\,\text{g22})\cos(\chi_m-\psi_m-\psi_n)+(i\,\text{g11}+\text{g22})\cos(\chi_m+\psi_m+\psi_n)+(i\,\text{g12}+\text{g21})\sin(\chi_m-\psi_m-\psi_n)+(\text{g21}-i\,\text{g12})\sin(\chi_m+\psi_m+\psi_n)))\Big)-$$
$$\text{Conjugate(h11)}\cos(\chi_n)\Big(2\cos(\psi_m+\psi_n)\Big(\text{g11}\Big(\frac{\text{g12 Conjugate(h12)}}{\text{g11 Conjugate(h11)}}-1\Big)\cos(\chi_m)+i\,\text{g22}\Big(\frac{\text{g21}}{\text{g22}}-\frac{\text{Conjugate(h12)}}{\text{Conjugate(h11)}}\Big)\sin(\chi_m)\Big)+$$
$$2\Big(\text{g11}\Big(\frac{\text{g12}}{\text{g11}}+\frac{\text{Conjugate(h12)}}{\text{Conjugate(h11)}}\Big)\cos(\chi_m)-i\,\text{g22}\Big(\frac{\text{g21 Conjugate(h12)}}{\text{g22 Conjugate(h11)}}+1\Big)\sin(\chi_m)\Big)\sin(\psi_m+\psi_n)\Big)\Big)$$

s_1 +

$$\frac{1}{2}\Big(\cos(\psi_m+\psi_n)\Big(\text{Conjugate(h11)}\cos(\chi_n)\Big(\text{g11}\Big(\frac{\text{g12}}{\text{g11}}+\frac{\text{Conjugate(h12)}}{\text{Conjugate(h11)}}\Big)\cos(\chi_m)-i\,\text{g22}\Big(\frac{\text{g21 Conjugate(h12)}}{\text{g22 Conjugate(h11)}}+1\Big)\sin(\chi_m)\Big)+$$
$$\text{Conjugate(h22)}\Big(i\,\text{g11}\Big(\frac{\text{g12 Conjugate(h21)}}{\text{g11 Conjugate(h22)}}+1\Big)\cos(\chi_m)+\text{g22}\Big(\frac{\text{g21}}{\text{g22}}+\frac{\text{Conjugate(h21)}}{\text{Conjugate(h22)}}\Big)\sin(\chi_m)\Big)\sin(\chi_n)\Big)-$$
$$\Big(\text{Conjugate(h11)}\cos(\chi_n)\Big(\text{g11}\Big(\frac{\text{g12 Conjugate(h12)}}{\text{g11 Conjugate(h11)}}-1\Big)\cos(\chi_m)+i\,\text{g22}\Big(\frac{\text{g21}}{\text{g22}}-\frac{\text{Conjugate(h12)}}{\text{Conjugate(h11)}}\Big)\sin(\chi_m)\Big)+$$
$$\text{Conjugate(h22)}\Big(i\,\text{g11}\Big(\frac{\text{g12}}{\text{g11}}-\frac{\text{Conjugate(h21)}}{\text{Conjugate(h22)}}\Big)\cos(\chi_m)-\text{g22}\Big(\frac{\text{g21 Conjugate(h21)}}{\text{g22 Conjugate(h22)}}-1\Big)\sin(\chi_m)\Big)\sin(\chi_n)\Big)\sin(\psi_m+\psi_n)\Big)$$

s_2 +

$$\frac{1}{2}\Big(\text{Conjugate(h22)}\sin(\chi_n)\Big(i\Big(i\,\text{g11}\Big(\frac{\text{g12}}{\text{g11}}+\frac{\text{Conjugate(h21)}}{\text{Conjugate(h22)}}\Big)\cos(\chi_m)+\text{g22}\Big(\frac{\text{g21 Conjugate(h21)}}{\text{g22 Conjugate(h22)}}+1\Big)\sin(\chi_m)\Big)\sin(\psi_m-\psi_n)-$$
$$\cos(\psi_m-\psi_n)\Big(\text{g11}\Big(\frac{\text{g12 Conjugate(h21)}}{\text{g11 Conjugate(h22)}}-1\Big)\cos(\chi_m)+i\,\text{g22}\Big(\frac{\text{g21}}{\text{g22}}-\frac{\text{Conjugate(h21)}}{\text{Conjugate(h22)}}\Big)\sin(\chi_m)\Big)\Big)+$$
$$\frac{1}{2}\,\text{Conjugate(h11)}\cos(\chi_n)\Big((i\,\text{g12}-\text{g21})\cos(\chi_m+\psi_m-\psi_n)+i\,\text{g12}\cos(\chi_m-\psi_m+\psi_n)+\text{g21}\cos(\chi_m-\psi_m+\psi_n)+$$
$$\frac{1}{\text{Conjugate(h11)}}(2\,\text{Conjugate(h12)}\,((i\,\text{g12}\cos(\chi_m)+\text{g22}\sin(\chi_m))\sin(\psi_m-\psi_n)-\cos(\psi_m-\psi_n)\,(i\,\text{g11}\cos(\chi_m)+\text{g21}\sin(\chi_m))))+$$
$$i\,\text{g11}\sin(\chi_m+\psi_m-\psi_n)+\text{g22}\sin(\chi_m+\psi_m-\psi_n)+(\text{g22}-i\,\text{g11})\sin(\chi_m-\psi_m+\psi_n)\Big)\Big)$$

s_3

■ References

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■ Appendix

■ Definitions of *Mathematica* functions for the model in this memo

```
<< i:\smatrifi\pestle.m

scatterMatrix = {{S11, S12}, {S21, S22}};

rotation[phi_] := {{Cos[phi], Sin[phi]}, {-Sin[phi], Cos[phi]}}

birefringent[psi_] := {{Cos[psi / 2] + I * Sin[psi / 2], 0}, {0, Cos[psi / 2] - I * Sin[psi / 2]}}
```

```
Simplify[rotation[-phi] . birefringent[psi] . rotation[phi]]
```

$$\left\{\left\{\text{Cos}\left[\frac{\text{psi}}{2}\right] + \text{I}\,\text{Cos}[2\,\text{phi}]\,\text{Sin}\left[\frac{\text{psi}}{2}\right],\ \text{I}\,\text{Sin}[2\,\text{phi}]\,\text{Sin}\left[\frac{\text{psi}}{2}\right]\right\},\right.$$
$$\left.\left\{\text{I}\,\text{Sin}[2\,\text{phi}]\,\text{Sin}\left[\frac{\text{psi}}{2}\right],\ \text{Cos}\left[\frac{\text{psi}}{2}\right] - \text{I}\,\text{Cos}[2\,\text{phi}]\,\text{Sin}\left[\frac{\text{psi}}{2}\right]\right\}\right\}$$

```
retardPlate[psi_, phi_, phase_] := Exp[-I * phase] * {{Cos[psi/2] + I Cos[2 phi] Sin[psi/2], I Sin[2 phi] Sin[psi/2]},
   {I Sin[2 phi] Sin[psi/2], Cos[psi/2] - I Cos[2 phi] Sin[psi/2]}}
```

```
xPolarizer = {{1, 0}, {0, 0}};
```

```
projection[phase_, chi_, delta_] :=
 Exp[-I * phase] * {{Cos[chi] + I Sin[2 delta] Sin[chi], -I Cos[2 delta] Sin[chi]}, {0, 0}}
```

```
parallactic[psi_] := {{Cos[psi], Sin[psi]}, {-Sin[psi], Cos[psi]}}
```

```
gainMatrix[gainx_, gainy_] := {{gainx, 0}, {0, gainy}}
```

```
General::spell1 : Possible spelling error: new symbol name "gainy" is similar to existing symbol "gainx".
```

```
feedErrorMatrix[dxy_, dyx_] := {{1, dxy}, {dyx, 1}}
```

```
feedMatrix[gainx_, gainy_, dxy_, dyx_] := gainMatrix[gainx, gainy] . feedErrorMatrix[dxy, dyx]
```

```
overallScatter[{p1_, p2_}, {chi1_, chi2_}, {delta1_, delta2_},
  {psi1_, psi2_}, {gainx1_, dxy1_, dyx1_, gainy1_}, {gainx2_, dxy2_, dyx2_, gainy2_}] :=
 {{E^-Ip1 (gainx1 Cos[chi1] (Cos[psi1] - dxy1 Sin[psi1]) - I Sin[chi1]
       (gainy1 Cos[2 delta1] (dyx1 Cos[psi1] - Sin[psi1]) + gainx1 Sin[2 delta1] (-Cos[psi1] + dyx1 Sin[psi1]))),
   E^-Ip1 (gainx1 Cos[chi1] (dxy1 Cos[psi1] + Sin[psi1]) - I Sin[chi1]
       (-gainx1 Sin[2 delta1] (dxy1 Cos[psi1] + Sin[psi1]) + gainy1 Cos[2 delta1] (Cos[psi1] + dyx1 Sin[psi1])))},
  {E^-Ip2 (gainx2 Cos[chi2] (Cos[psi2] - dxy2 Sin[psi2]) - I Sin[chi2]
       (gainy2 Cos[2 delta2] (dyx2 Cos[psi2] - Sin[psi2]) + gainx2 Sin[2 delta2] (-Cos[psi2] + dxy2 Sin[psi2]))),
   E^-Ip2 (gainx2 Cos[chi2] (dxy2 Cos[psi2] + Sin[psi2]) - I Sin[chi2]
       (-gainx2 Sin[2 delta2] (dxy2 Cos[psi2] + Sin[psi2]) + gainy2 Cos[2 delta2] (Cos[psi2] + dyx2 Sin[psi2])))}}
```

```
General::spell1 : Possible spelling error: new symbol name "dyx1" is similar to existing symbol "dxy1".

General::spell1 : Possible spelling error: new symbol name "gainy1" is similar to existing symbol "gainx1".

General::spell1 : Possible spelling error: new symbol name "dyx2" is similar to existing symbol "dxy2".

General::stop : Further output of General::spell1 will be suppressed during this calculation.
```

```
crossCorrelation[{{s11_, s12_}, {s21_, s22_}}] :=
 {1/2 (s11 Conjugate[s21] + s12 Conjugate[s22]), 1/2 (s11 Conjugate[s21] - s12 Conjugate[s22]),
  1/2 (s12 Conjugate[s21] + s11 Conjugate[s22]), 1/2 I (s12 Conjugate[s21] - s11 Conjugate[s22])}

autoCorrelation[{s11_, s12_}] := {1/2 (Abs[s11]^2 + Abs[s12]^2), 1/2 (Abs[s11]^2 - Abs[s12]^2),
  1/2 (s12 Conjugate[s11] + s11 Conjugate[s12]), 1/2 I (s12 Conjugate[s11] - s11 Conjugate[s12])}
```

```
autoCorrelator[{{g11_, g12_}, {g21_, g22_}}, {chi_, psi_}] :=
 {1/2 (g11 Conjugate[g11] (1 + g12 Conjugate[g12]) Cos[chi]^2 + g22 (1 + g21 Conjugate[g21]) Conjugate[g22] Sin[chi]^2 -
     1/2 I (g22 Conjugate[g11] (g21 + Conjugate[g12]) - g11 (g12 + Conjugate[g21]) Conjugate[g22]) Sin[2 chi]),
  1/4 (Cos[2 psi] (-2 g11 Conjugate[g11] (-1 + g12 Conjugate[g12]) Cos[chi]^2 +
        2 g22 (-1 + g21 Conjugate[g21]) Conjugate[g22] Sin[chi]^2 +
        I (g22 Conjugate[g11] (-g21 + Conjugate[g12]) + g11 (-g12 + Conjugate[g21]) Conjugate[g22]) Sin[2 chi]) -
     (2 g11 Conjugate[g11] (g12 + Conjugate[g12]) Cos[chi]^2 + 2 g22 (g21 + Conjugate[g21]) Conjugate[g22] Sin[chi]^2 -
        I (Conjugate[g11] (g22 + g21 g22 Conjugate[g12]) - g11 (1 + g12 Conjugate[g21]) Conjugate[g22]) Sin[2 chi])
      Sin[2 psi]),
  1/4 (Cos[2 psi]
      (2 g11 Conjugate[g11] (g12 + Conjugate[g12]) Cos[chi]^2 + 2 g22 (g21 + Conjugate[g21]) Conjugate[g22] Sin[chi]^2 -
        I (Conjugate[g11] (g22 + g21 g22 Conjugate[g12]) - g11 (1 + g12 Conjugate[g21]) Conjugate[g22]) Sin[2 chi]) -
     (2 g11 Conjugate[g11] (-1 + g12 Conjugate[g12]) Cos[chi]^2 -
        2 g22 (-1 + g21 Conjugate[g21]) Conjugate[g22] Sin[chi]^2 +
        I (g22 Conjugate[g11] (g21 - Conjugate[g12]) + g11 (g12 - Conjugate[g21]) Conjugate[g22]) Sin[2 chi])
      Sin[2 psi]),
  1/4 (2 I (g11 Conjugate[g11] (g12 - Conjugate[g12]) Cos[chi]^2 + g22 (-g21 + Conjugate[g21]) Conjugate[g22] Sin[chi]^2) -
     (g22 Conjugate[g11] (-1 + g21 Conjugate[g12]) + g11 (-1 + g12 Conjugate[g21]) Conjugate[g22]) Sin[2 chi])}
```

```
crossCorrelator[{{g11_, g12_}, {g21_, g22_}},
  {{h11_, h12_}, {h21_, h22_}}, {chi1_, chi2_}, {psi1_, psi2_}] := {2 Cos[psi1 - psi2]
     (Conjugate[h11] Cos[chi2] (g11 (1 + g12 Conjugate[h12]) Cos[chi1] - I g22 (g21 + Conjugate[h12]) Sin[chi1]) +
       Conjugate[h22] (I g11 (g12 + Conjugate[h21]) Cos[chi1] + g22 (1 + g21 Conjugate[h21]) Sin[chi1]) Sin[chi2]) -
    2 (Conjugate[h11] Cos[chi2] (g11 (g12 - Conjugate[h12]) Cos[chi1] + I g22 (-1 + g21 Conjugate[h12]) Sin[chi1]) +
       Conjugate[h22] (I g11 (-1 + g12 Conjugate[h21]) Cos[chi1] + g22 (-g21 + Conjugate[h21]) Sin[chi1]) Sin[chi2])
     Sin[psi1 - psi2],
   -Conjugate[h11] Cos[chi2]
     (2 Cos[psi1 + psi2] (g11 (-1 + g12 Conjugate[h12]) Cos[chi1] + I g22 (g21 - Conjugate[h12]) Sin[chi1]) +
       2 (g11 (g12 + Conjugate[h12]) Cos[chi1] - I g22 (1 + g21 Conjugate[h12]) Sin[chi1]) Sin[psi1 + psi2]) +
    Conjugate[h22] Sin[chi2] (-2 I g11 Cos[chi1] (g12 Cos[psi1 + psi2] + Sin[psi1 + psi2]) -
       2 g22 Sin[chi1] (Cos[psi1 + psi2] + g21 Sin[psi1 + psi2]) +
       Conjugate[h21] (I (g11 + I g22) Cos[chi1 - psi1 - psi2] + (I g11 + g22) Cos[chi1 + psi1 + psi2] +
          (I g11 g12 + g21 g22) Sin[chi1 - psi1 - psi2] + (-I g11 g12 + g21 g22) Sin[chi1 + psi1 + psi2])),
   2 (Cos[psi1 + psi2]
       (Conjugate[h11] Cos[chi2] (g11 (g12 + Conjugate[h12]) Cos[chi1] - I g22 (1 + g21 Conjugate[h12]) Sin[chi1]) +
         Conjugate[h22] (I g11 (1 + g12 Conjugate[h21]) Cos[chi1] + g22 (g21 + Conjugate[h21]) Sin[chi1]) Sin[chi2]) -
      (Conjugate[h11] Cos[chi2]
          (g11 (-1 + g12 Conjugate[h12]) Cos[chi1] + I g22 (g21 - Conjugate[h12]) Sin[chi1]) + Conjugate[h22]
          (I g11 (g12 - Conjugate[h21]) Cos[chi1] - g22 (-1 + g21 Conjugate[h21]) Sin[chi1]) Sin[chi2]) Sin[psi1 + psi2]),
   2 (Conjugate[h22] Sin[chi2]
       (-Cos[psi1 - psi2] (g11 (-1 + g12 Conjugate[h21]) Cos[chi1] + I g22 (g21 - Conjugate[h21]) Sin[chi1]) +
         I (I g11 (g12 + Conjugate[h21]) Cos[chi1] + g22 (1 + g21 Conjugate[h21]) Sin[chi1]) Sin[psi1 - psi2]) +
       1/2 Conjugate[h11] Cos[chi2] ((I g11 g12 - g21 g22) Cos[chi1 + psi1 - psi2] +
          I g11 g12 Cos[chi1 - psi1 + psi2] + g21 g22 Cos[chi1 - psi1 + psi2] + 2 Conjugate[h12] (-Cos[psi1 - psi2]
              (I g11 Cos[chi1] + g21 g22 Sin[chi1]) + (I g11 g12 Cos[chi1] + g22 Sin[chi1]) Sin[psi1 - psi2]) +
          I g11 Sin[chi1 + psi1 - psi2] + g22 Sin[chi1 + psi1 - psi2] + (-I g11 + g22) Sin[chi1 - psi1 + psi2])) } /
  4
```